Economic Growth and International Trade: The Dixit-Norman Approach

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Abstract

This paper analyzes the transitional dynamics of the open-economy versions of the growth models of Solow [47], Romer [41], Lucas [35], Romer [42], Grossman and Helpman [24] (Chapters 3 and 4), Jones [27], and Segerstrom [45], among others. Rather than addressing the problem directly, we prove that the Dixit-Norman proposition, that the world economy replicates the equilibrium of the hypothetical integrated world economy under certain conditions, holds true in a broad class of open-economy growth models, which includes the ones just mentioned. Consequently, the world economy's transitional dynamics are identical to the integrated economy's. Since results for the latter exist in the literature, all that remains to be done is determine which parts of world-wide economic activity are conducted where. Another important implication of the Dixit-Norman result for growth models is that the question of whether international economic integration speeds up growth boils down to the question of whether larger economies grow faster (i.e., whether scale effects prevail).

JEL classification: F12, F15, O41

Key words: economic growth, international trade, international knowledge spillovers, transitional dynamics, multinational corporations, international patent licensing, scale effects.

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1 Introduction

"The greatest task for economists is to develop a consistent body of theory that explains the relationship between trade and growth."
Lewer and van den Berg [32] (p. 390).

Economists' understanding of how a world economy with free trade in products and internationally immobile factors of production functions has been sharpened significantly by the seminal contribution of Dixit and Norman [17]. Dixit and Norman's central insight in [17] (Chapter 4) is that the equilibrium of a Heckscher-Ohlin-type world economy replicates the equilibrium of a hypothetical integrated economy, defined by the complete absence of national borders, under certain conditions (technologies are the same in all countries and display constant returns to scale, and competitive pricing behavior).[1] Helpman and Krugman in [26] extended the Dixit-Norman methodology to settings with imperfect competition in product markets. And Grossman and Helpman [23], [24] (Chapter 7) recast the approach in a dynamic setting in order to study the dynamic evolution of comparative advantage in R&D growth models. The aim of the present paper is to extend the Dixit-Norman approach to a much broader class of growth models. Specifically, we consider a (class of) model(s) which is general in at least two respects. First, it encompasses, as special cases, the models of Solow [47], Romer [41], Lucas [35], Romer [42], Grossman and Helpman [24] (Chapters 3 and 4), Jones [27], and Segerstrom [45], among others.[2] Second, it incorporates the central mechanisms driving economic growth, viz. capital accumulation, learning by doing, human capital accumulation, product innovation, and quality upgrading, as well as combinations of two or more of these mechanisms. We show that, in this general model, *if tastes*[3] *and technologies are the same in all countries, if the productive activities are characterized by constant returns to scale with respect to the private factors of production, and if spillover effects are international in scope, then the replication of the equilibrium of the hypothetical*

[1]This approach was initiated by Travis [52] (Chapter 2). Since Dixit and Norman [17] (Chapter 4) made it popular, it is justified to name it after them. They remark that Samuelson [44] already "saw through the whole problem, and we think that if he had filled out some of the asides and terse remarks he makes, he would have developed the argument much as we have done here" ([17], p. 125). An empirical argument against the Dixit-Norman approach is McCallum's puzzle of home bias in trade ([36], [39], Section 2).

[2]The selection of models covered as special cases does not reflect our relative appreciation of different growth models. Many equally important models cannot be subsumed under the general model developed here. Because of the sheer size of the literature that has been developing over the past fifteen years, it appears pointless to try to list the most important contributions left out here. Nonetheless, in view of the list of models included as special cases, it seems justified to claim that the model is general.

[3]The assumption of identical tastes is inessential in the static theories but essential in the dynamic models analyzed in Grossman and Helpman [23], [24] (Chapter 7) and here.

integrated economy with free mobility of all factors tends to be possible if either physical capital is mobile internationally (or absent from the model) or the countries' relative factor endowments are sufficiently similar. The presence of multinational corporations or international patent licensing is a sufficient condition in the former case and is conducive to replicability in the latter case.

The most important implication of this theorem is that it allows a fruitful fresh look at the transitional dynamics of open-economy endogenous growth models. Suppose a model satisfies the conditions of our Dixit-Norman theorem. Then, rather than analyzing the transitional dynamics of the world economy directly, we can proceed in two steps. First, since the world economy replicates the integrated economy, we immediately obtain results about the behavior of world aggregates along transition paths by applying existing results about the transitional dynamics of closed-economy growth models. All that remains to be done is characterize which portions of world-wide economic activity are conducted where. Proceeding this way, this paper derives original results about the transitional dynamics of the open-economy versions of some of the most important growth models (with no need to solve a single differential equation) and provides a systematic restatement of existing results for other models.

Moreover, the finding that the world economy replicates the hypothetical integrated economy has direct significance for the question of whether international economic integration has an impact on economies' long-run growth performance. To see this, suppose the assumptions of the proposition are satisfied and compare the development path of a country in complete economic isolation, on the one hand, with that of a country with international trade and international spillovers, on the other hand. Given that the world economy replicates the hypothetical integrated equilibrium, the question of whether the economy (and the rest of the world) grows faster in the long run in the second case is equivalent to the question of whether the integrated equilibrium features scale effects (i.e., if a greater workforce implies faster long-run growth).[4]

We make several simplifying assumptions in order to keep the model as simple as is consistent with the aim that it covers some prominent growth models and incorporates the most important engines of growth. First, there is only one final good. It would be straightforward to treat the case of two goods following Grossman and Helpman's analysis of dynamic comparative advantage in [24] (Chapter 7). Since the model includes different economic activities with possibly differing factor intensities, there is scope for Heckscher-Ohlin-like trade patterns anyway. Introducing a second consumption good would expand the opportunities for trade caused by differences in relative factor endowments. Second, the final good is homogeneous. Product innovation does not yield new differentiated consumer goods, but

[4]Many standard growth models have the scale effect property (see below). The literature on growth without scale effects has been pioneered by Jones [27], Young [55], Segerstrom [45], and Arnold [2].

new intermediate goods used in the production of final goods. Similarly, quality upgrading takes place in the intermediate goods sector. It would be straightforward to restate the model with R&D aimed at consumer goods rather than at intermediate production goods (cf. [24], Chapters 3 and 4). Third, there is only one R&D activity: either product innovation or quality upgrading. Originating from Young (1998), models with both kinds of R&D have been analyzed. Recent work by Li [33] shows that two-R&D-sector models tend to behave similarly to one-R&D-sector models without scale effects. These are covered as a special case in our analysis. Fourth, labor is homogeneous. The outcome of human capital accumulation is not a different kind of high-skilled labor supply (as, for example, in [24], Chapter 7), but an additional supply of the homogeneous labor. So the model is silent on issues relating to wage dispersion. In order for the Dixit-Norman approach to work, we need three further assumptions. First, as in the static Dixit-Norman analysis, returns to scale in the private factors of production are constant in all sectors. Kortum [29] and Stokey [49] discuss the implications of decreasing returns in the private factors of production in R&D. By contrast, we do not put a restriction on social returns to scale in final goods production and R&D. Eicher and Turnovsky [18] carefully investigate the requirements which have to be placed on social returns in order for balanced growth to be possible. Second, countries have identical tastes and technologies. They are "similar" except possibly in size. So the model is silent on issues of North-South trade between more- and less-developed countries (see [24], Chapters 11 and 12, [25], [4]). Third, we assume that spillover effects are international in scope, as knowledge flows freely across national borders (cf. [20]). Empirical support for this assumption is provided by Coe and Helpman [14] (see also [34]). Grossman and Helpman in [24] (Chapters 8 and 9) and Rivera-Batiz and Romer in [40] show that the scope of knowledge flows is possibly important for the determination of equilibrium growth rates and the models' dynamics.

Section 2 presents the model. Section 3 describes the equilibrium in the integrated economy. Section 4 derives the conditions under which the world economy with international trade and international spillovers, but without labor mobility, replicates the integrated-economy equilibrium. Section 5 focuses on several prominent special cases of the model. Original results concerning the models' transitional dynamics are derived and the issue of whether international economic integration boosts growth are discussed. Section 6 concludes.

2 Model

This section states the assumptions about technologies, tastes, and market structure underlying our general model. For now, we ignore the presence of national borders which inhibit factor movements, so that our focus is on the hypothetical integrated economy. The backbone of the model is formed by production functions for final output, intermediate products, R&D, and human capital which display constant returns to scale in the private factors of production (we allow for the special case that the output levels are identically zero for all inputs, which means that the corresponding economic activity is not performed).[5]

A homogeneous final good, which can be used for consumption or investment (depreciation of capital is ignored), is produced by perfectly competitive firms according to the neoclassical constant-returns-to-scale production function F_Y:[6]

$$Y = F_Y(K_Y, B_Y L_Y, D_Y), \tag{1}$$

where Y is aggregate output, K_Y and L_Y are capital and labor input per firm, respectively, B_Y is a labor-augmenting productivity parameter common to all firms, and D_Y is an index of intermediate goods inputs explained below. The productivity parameter, B_Y, obeys

$$B_Y = \frac{K_Y^\eta}{L_Y^\varepsilon}. \tag{2}$$

This is the usual way to capture learning-by-doing effects in the production of final goods. Later on, we will allow for five different specifications. First, there is no learning by doing: $\eta = \varepsilon = 0$, $B_Y = 1$. Second, externalities emanate from the capital stock with diminishing returns: $0 < \eta < 1$ and $\varepsilon = 0$, so that $B_Y = K_Y^\eta$. Third, externalities emanate from the capital stock with non-diminishing returns: $\eta = 1$, $\varepsilon = 0$, $B_Y = K_Y$. Fourth, externalities emanate from the capital intensity with non-diminishing returns: $\eta = \varepsilon = 1$, $B_Y = K_Y/L_Y$. Fifth, positive externalities emanate from L_Y: $\eta = 0$, $\varepsilon < 0$. For the sake of simplicity, we ignore learning-by-doing effects in the other sectors of the economy.

The model comprises a *product variety (PV)* variant and a *quality upgrading (QU)* variant, which differ with regard to how D_Y is produced from a set of intermediates. The input of quality ω of intermediate j is denoted $X_\omega(j)$. In the PV case, the number of producible intermediates is denoted as A, and only one quality ω of each product exists, so we can drop index ω and write $X(j)$ for the input of intermediate j. More generally, we adopt the convention that subscript ω is dropped in formulas which apply to the PV variant of the model. The production function reads:

$$D_Y = \left[\int_0^A X(j)^\alpha dj\right]^{\frac{1}{\alpha}}. \tag{3}$$

Returns to scale are constant. The constant elasticity of substitution between any pair j and j' of intermediates is $-1/(1-\alpha)$ $(0 < \alpha < 1)$. In the QU model, the number of producible intermediates

[5] In our model, as always, the Dixit-Norman replication argument relies on exactly constant returns to scale. The idea is that if this knife-edge assumption is violated slightly, the world economy behaves similarly to the integrated economy.

[6] We ignore the time argument wherever this does not cause confusion.

is constant ($A = 1$), but different qualities ω of the given set of intermediates j are available. The highest quality producible of intermediate ω is denoted $\Omega(j)$. The production function for D_Y is:

$$D_Y = \exp\left\{\int_0^1 \log\left[\sum_{\omega=1}^{\Omega(j)} \lambda^\omega X_\omega(j)\right] dj\right\}. \tag{4}$$

Returns to scale are constant. For all intermediates j, one unit of quality $\omega + 1$ is a perfect substitute for λ (> 1) units of quality ω. The elasticity of substitution between two intermediates j and j' of given qualities is -1. For the sake of simplicity, we assume that the intermediates are not used directly in the other sectors of the economy.[7] Both in the PV, and the QU, variants of the model, the intermediates are produced from capital $K_{X_\omega}(j)$ and labor $L_{X_\omega}(j)$ according to the neoclassical constant-returns-to-scale production function

$$X_\omega(j) = F_X[K_{X_\omega}(j), L_{X_\omega}(j)]. \tag{5}$$

R&D increases the number of producible intermediates, A, in the PV model and upgrades the highest qualities producible $\Omega(j)$ in the QU model. Let K_A and L_A denote the aggregate capital and labor inputs in R&D and $F_A(K_A, L_A)$ a neoclassical constant-returns-to-scale production function. In the PV model, the number of new intermediates, $\dot{A}$, is $F_A(K_A, L_A)A^\chi$ ($\chi \leq 1$). The presence of the aggregate number of currently producible intermediates, A, in the R&D production function reflects the presence of knowledge spillovers. In the QU model, let $I(j)\,dt$ denote the probability of a quality jump (the rate of innovation) in industry j in the short time interval dt. We assume that the amount of R&D targeted at each market j is the same ($I(j) \equiv I$), so the number of markets with a quality improvement in a short time interval dt is $d[\int_0^1 \Omega(j)dj] = I\,dt$. The rate of innovation is $I = F_A(K_A, L_A)A^{-(1-\chi)}$, where $A(t) = \exp[\int_{-\infty}^t I(\tau)d\tau]$. The presence of the cumulated past aggregate innovation rates, $I(\tau)$, captures the effect that successes in R&D become harder and harder to accomplish. Differentiating the expression for $A(t)$ and inserting the equation for the innovation rate, I, gives $\dot{A} = AI = F_A(K_A, L_A)A^\chi$. So

$$\dot{A} = F_A(K_A, L_A)A^\chi \tag{6}$$

holds true both in the PV model and in the QU model. In the latter, we have, from $d[\int_0^1 \Omega(j)dj] = I\,dt$ and $I = \dot{A}/A$,

$$\frac{d}{dt}\left[\int_0^1 \Omega(j)dj\right] = I = \frac{\dot{A}}{A}. \tag{7}$$

Due to technological leadership or protection of intellectual property rights, the innovator of a new product variety or of a new quality is the only supplier of the respective variety or quality, respectively.

[7] They are used indirectly because, as will be explained below, other sectors use physical capital produced in the final goods sector.

It is assumed that at the outset there is also only one supplier of each intermediate in the PV model and only one supplier of the highest quality producible of each intermediate in the QU case.
The economy is populated by a continuum of unit length of identical Barrovian families which share the total family consumption equally among the family members. Letting c denote per-capita consumption, the utility function is $\int_t^\infty [c(\tau)^{1-\sigma}/(1-\sigma)]e^{-\rho(\tau-t)}d\tau$ $(\rho, \sigma > 0)$[8]. The total population size is denoted N and grows at rate g_N (≥ 0). The economy is said to display scale effects if $g_N = 0$ and N positively affects the steady-state growth rate of per-capita output and consumption. Each agent supplies l units of labor, so aggregate labor supply is Nl. The initial endowment $l(0)$ is identical for all agents. People can increase their effective per-capita supply of labor by acquiring human capital in education. Let l_l denote the labor devoted to human capital accumulation, l'_l the input of other agents' (teachers) labor in the education process, and k_l capital per student. Then

$$\dot{l} = F_l(k_l, l_l, l'_l), \tag{8}$$

where F_l is a constant-returns-to-scale production function.[9]
All markets except those for the intermediate goods are perfectly competitive and clear in equilibrium. In the PV model, the producers of the intermediates are monopolistic competitors. In the QU model, the intermediate-goods producers are engaged in price competition. There is free entry into R&D.

3 Integrated Economy

In this section, we characterize the equilibrium of the hypothetical integrated world economy described above. The equilibrium conditions describe the behavior of producers and households and market clearing.

3.1 Producers

In the final goods sector, price equals unit cost due to perfect competition. The firms' cost minimization problem can be split into two stages. Stage one: minimize the cost, p_D, of producing one unit of D_Y given (3) or (4). Stage two: minimize total cost given (1) and p_D. For the PV model, the first stage consists of minimizing the cost, $\int_0^A p(j)X(j)dj$, of producing one unit of D_Y subject to (3), where

[8]It is understood that instantaneous utility is logarithmic if $\sigma = 1$.

[9]We treat teachers' labor as a private good here. Alternatively, it can be assumed that, up to a certain level z (> 0) of students, l'_l is non-rival and that, beyond that level, l'_l is unproductive. Suppose students form coalitions of size z and share the wage costs for their teacher uniformly, and normalize $z = 1$. Then the analysis proceeds analogously.

$p(j)$ is the price of intermediate j. (cf. [24], Chapter 3). The solution to this problem yields the input coefficients

$$a_{X(j)} = \frac{p(j)^{-\frac{1}{1-\alpha}}}{\left[\int_0^A p(j')^{-\frac{\alpha}{1-\alpha}} dj'\right]^{\frac{1}{\alpha}}}$$

and the price of one unit of D_Y:

$$p_D = \left[\int_0^A p(j)^{-\frac{\alpha}{1-\alpha}} dj\right]^{-\frac{1-\alpha}{\alpha}}.$$

In the QU model, the first stage of the cost minimization problem entails minimizing $\int_0^1 \sum_{\omega=1}^{\Omega(j)} p_\omega(j) X_\omega(j) dj$ subject to (4) with $D_Y = 1$, where $p_\omega(j)$ is the price of quality ω of intermediate j (cf. [24], Chapter 4). The solution to this problem entails the input coefficients

$$a_{X_\omega(j)} = \begin{cases} \frac{p_D}{p_\omega(j)}; & \text{for } \omega = \tilde{\omega}(j) \\ 0; & \text{for } \omega \neq \tilde{\omega}(j), \end{cases}$$

where $\tilde{\omega}(j) \equiv \arg\min_\omega \{p_\omega(j)/\lambda^\omega\}$, and the price of one unit of D_Y:

$$p_D = \exp\left\{\int_0^1 \left[\log p_{\tilde{\omega}(j)}(j) - \tilde{\omega}(j) \log \lambda\right] dj\right\}.$$

Turning to the second stage of the cost minimization problem, let w and r denote the wage rate and the interest rate, respectively. The total cost, $rK_Y + wL_Y + p_D D_Y$, of producing $Y = 1$ is minimized subject to (1). This gives the input coefficients $a_{K_Y}(r, w, p_D, B_Y)$, $a_{L_Y}(r, w, p_D, B_Y)$, and $a_{D_Y}(r, w, p_D, B_Y)$ of capital, labor, and D_Y, respectively, and the unit cost function $c_Y(r, w, p_D, B_Y)$. Choosing the final good as the numeraire, competitive pricing implies:

$$1 = c_Y(r, w, p_D, B_Y). \tag{9}$$

The final goods sector's demand for intermediates is $a_{X_\omega(j)} a_{D_Y}(r, w, p_D, B_Y) Y$. Furthermore, (2) can be rewritten as

$$B_Y = \frac{[a_{K_Y}(r, w, p_D, B_Y) Y]^\eta}{[a_{L_Y}(r, w, p_D, B_Y) Y]^\varepsilon}. \tag{10}$$

Next, consider the producers of intermediate goods. Minimizing the cost $rK_{X_\omega}(j) + wL_{X_\omega}(j)$ subject to (5) yields the input coefficients $a_{K_{X_\omega}(j)}(r, w)$ and $a_{L_{X_\omega}(j)}(r, w)$ and the unit cost function $c_x(r, w)$. The single producer of an intermediate good j (PV) or a given quality ω of an intermediate good j (QU) maximizes his monopoly profit $\pi_\omega(j) \equiv [p_\omega(j) - c_x(r, w)] X_\omega(j)$ given the final goods sector's demand. In the PV model, the price elasticity of demand is $-1/(1-\alpha)$. So intermediate goods producers maximize profit with the markup price $c_x(r, w)/\alpha$. In the QU model, for each good j, only the producer $\tilde{\omega}(j)$ with the lowest quality-adjusted price, $p_\omega(j)/\lambda^\omega$, faces a positive demand. This producer's price elasticity of demand is -1, so his profits increase as he raises his price. In equilibrium, the producer of the maximum-quality intermediate, $\Omega(j)$, prices the lower-quality producers out of the market ($\tilde{\omega}(j) = \Omega(j)$) with the limit price $\lambda c_x(r, w)$. Both in the PV variant and in the QU variant of the model, the monopoly prices, p, charged by the active producers are identical:

$$p = \mu c_x(r, w), \tag{11}$$

where $\mu \equiv 1/\alpha$ in the PV model and $\mu \equiv \lambda$ in the QU model. This has several important consequences. Since the demand curves $a_{X_{\omega(j)}} a_{D_Y}(r, w, p_D, B_Y)Y$ are also the same for each producer, so are the quantities brought out, X, and monopoly profits, π:

$$\pi = \left(1 - \frac{1}{\mu}\right) pX. \tag{12}$$

For the sake of notational convenience, we introduce the dummy variable $\mathcal{D}$ which equals zero for the PV model and unity for the QU model.[10]. Then p_D can be rewritten as:

$$p_D = p \left(A^{-\frac{1-\alpha}{\alpha}}\right)^{1-\mathcal{D}} \left\{\exp\left[-\log\lambda \int_0^1 \Omega(j)dj\right]\right\}^{\mathcal{D}} \tag{13}$$

and the input coefficients $a_{X_{\omega(j)}} \equiv a_X$ as:

$$a_X = \frac{p_D}{A^{1-\mathcal{D}} p}.$$

Finally, consider firms engaged in R&D. The reward to investments in R&D is the expected present value of the ensuing monopoly profits,

$$v(t) \equiv \int_t^\infty \exp\left\{-\int_t^\tau \left[r(\vartheta) + \mathcal{D}\frac{\dot{A}(\vartheta)}{A(\vartheta)}\right] d\vartheta\right\} \pi(\tau)d\tau. \tag{14}$$

The term $\exp\{-\int_t^\tau [\dot{A}(\vartheta)/A(\vartheta)]d\vartheta\}$ represents the probability of not losing a monopoly between t and τ in the QU model (since the instantaneous probability of losing a monopoly is $I = \dot{A}/A$) and acts like an additional discount factor. Let $c_A(r, w)$ denote the cost of producing $F(K_A, L_A) = 1$ and $a_{K_A}(r, w)$ and $a_{L_A}(r, w)$ the corresponding input coefficients. Suppose a firm uses $a_{K_A}(r, w)$ units of capital and employs $a_{L_A}(r, w)$ workers in R&D. This costs $c_A(r, w)$. In the PV model, the result is A^χ new intermediates, each worth v. So free entry into R&D implies $A^\chi v = c_A(r, w)$. In the QU model, the result is the innovation rate $A^{-(1-\chi)}$, and free entry implies $A^{-(1-\chi)} v = c_A(r, w)$. Hence,

$$A^{\chi-\mathcal{D}} v = c_A(r, w). \tag{15}$$

[10]So if a term Z_{PV} appears in the PV model but not in the QU model, writing $Z_{PV}^{1-\mathcal{D}}$ covers both cases. If Z_{QU} appears in the QU model where a term Z_{PV} appears in the PV model, we can write $Z_{PV}^{1-\mathcal{D}} Z_{QU}^{\mathcal{D}}$

3.2 Households

The households choose their investments in education k_l, l_l, and l_l' and per-capita consumption c so as to maximize the household's members' utility. This problem can be solved in two stages. In the first stage, the cost, $rk_l + w(l_l + l_l')$, of producing one unit of $F_l(k_l, l_l, l_l')$ is minimized. This yields the cost function $c_l(r,w)$ and the input coefficients $a_{k_l}(r,w)$, $a_{l_l}(r,w)$, and $a_{l_l'}(r,w)$. In the second stage, the households solve:

$$\max_{c,\dot{l}} : \int_t^\infty \frac{c(\tau)^{1-\sigma}}{1-\sigma} e^{-\rho(\tau-t)} d\tau$$
$$\text{s.t.: } \dot{\nu} = r\nu + wl - c_l(r,w)\dot{l} - c,$$

where ν is the household's per-capita financial wealth. The current-value Hamiltonian for this problem is

$$\mathcal{H} \equiv \frac{c^{1-\sigma}}{1-\sigma} + \zeta_\nu[r\nu + wl - c_l(r,w)\dot{l} - c] + \zeta_l \dot{l}$$

with ζ_ν and ζ_l as co-state variables. The necessary and sufficient conditions for an interior optimum are:

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\sigma} - \zeta_\nu = 0$$
$$\dot{\zeta}_\nu = \rho\zeta_\nu - \frac{\partial \mathcal{H}}{\partial \nu} = \rho\zeta_\nu - r\zeta_\nu$$
$$\frac{\partial \mathcal{H}}{\partial \dot{l}} = -\zeta_\nu c_l(r,w) + \zeta_l = 0$$
$$\dot{\zeta}_l = \rho\zeta_l - \frac{\partial \mathcal{H}}{\partial l} = \rho\zeta_l - \zeta_\nu w$$

plus two transversality conditions. The former two conditions yield $-\sigma\dot{c}/c = \dot{\zeta}_\nu/\zeta_\nu$, $\dot{\zeta}_\nu/\zeta_\nu = \rho - r$, and, hence, the Ramsey rule:

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\sigma}. \tag{16}$$

The third condition implies $\dot{c}_l/c_l = \dot{\zeta}_l/\zeta_l - \dot{\zeta}_\nu/\zeta_\nu$. Substituting $\dot{\zeta}_l/\zeta_l = \rho - \zeta_\nu w/\zeta_l = \rho - w/c_l$ (from the third and fourth conditions) and $\dot{\zeta}_\nu/\zeta_\nu = \rho - r$ yields

$$\dot{c}_l(r,w) = rc_l(r,w) - w.^{11} \tag{17}$$

[11] The consumer side can easily be generalized in at least two respects. First, we can allow for uneven distributions of financial wealth across agents. Second, the assumption that all agents start with the same amount of initial human capital is dispensable. Neither generalization would affect Eqs. (16) and (17). Moreover, the education technology (8) could be replaced with a Findlay-Kierzkowski [22] education technology, which specifies the amount of human capital per capita, $l(S)$, obtained with S periods in education ($l(S)$ is twice differentiable with $l'(S) > 0 > l''(S)$). With finite lives of length T (> 0), the necessary and sufficient condition for an interior optimum education spell, S, for individuals born at t is: $\exp[-\int_t^{t+S} r(\vartheta)d\vartheta]w(t+S)l(S) = \int_{t+S}^{t+T} \exp[-\int_t^\tau r(\vartheta)d\vartheta]w(\tau)l'(S)d\tau$. This equation would replace equation (17). Grossman and Helpman [24] (Section 5.2) and Dinopoulos and Segerstrom [16] pursue this approach. For our purposes, this approach is less fruitful, since it does not allow the analysis of transitional dynamics.

3.3 Market clearing

It remains for us to formulate the market clearing conditions. Equality of supply and demand in the market for final goods requires

$$Y = \dot{K} + Nc. \tag{18}$$

The demand for intermediates is $a_X a_{D_Y}(r, w, p_D, B_Y)Y$. Using the fact that $a_X = p_D/(A^{1-\mathcal{D}}p)$, the condition for an equilibrium in the markets for the intermediates can be written as:

$$X = \frac{p_D}{A^{1-\mathcal{D}}p} a_{D_Y}(r, w, p_D, B_Y)Y. \tag{19}$$

The markets for physical capital and labor clear if

$$\begin{pmatrix} K \\ Nl \end{pmatrix} = \begin{pmatrix} a_{K_Y}(r, w, p_D, B_Y) & a_{K_X}(r, w) & a_{K_A}(r, w) \\ a_{L_Y}(r, w, p_D, B_Y) & a_{L_X}(r, w) & a_{L_A}(r, w) \end{pmatrix} \begin{pmatrix} Y \\ A^{1-\mathcal{D}}X \\ A^{1-\chi}\left(\frac{\dot{A}}{A}\right)^{1-\mathcal{D}} I^{\mathcal{D}} \end{pmatrix}$$
$$+ \begin{pmatrix} a_{k_l}(r, w) \\ a_{l_l}(r, w) + a_{l'_l}(r, w) \end{pmatrix} N\dot{l}. \tag{20}$$

The four terms on the right-hand side of these two equations are capital and labor, respectively, in their four different uses, i.e. final goods production, intermediate goods production, R&D, and education. For instance, capital in R&D is $K_A = a_{K_A}(r, w)\dot{A}/A^{\chi}$ in the PV model (i.e. for $\mathcal{D} = 0$) and $K_A = a_{K_A}(r, w)A^{1-\chi}I$ in the QU model (i.e. for $\mathcal{D} = 1$). An equilibrium in the market for financial capital (i.e., $N\nu = K + Av$ with $A = 1$ in the QU model) prevails if the other markets are in equilibrium.

3.4 Equilibrium

For the PV model (i.e., for $\mathcal{D} = 0$), Eqs. (9)-(20) form a system of 13 equations in the 13 variables B_Y, r, w, p_D, p, X, π, A, v, c, Y, K, and l. For the QU model ($\mathcal{D} = 1$), the additional variables $\int_0^1 \Omega(j)dj$ and I appear in (13) and (20), respectively, and (7) provides two additional equations. So we have a determinate system of equations.[12] With appropriate assumptions on the parameters of the model and the specific functional forms, a solution exists, so that (9)-(20) determine the evolution of the integrated economy through time. The validity of the system is not confined to balanced-growth paths. We refer to the equilibrium of the integrated economy as the integrated equilibrium.

[12] The reason why, other than in static models, identical tastes are required for the validity of the Dixit-Norman theorem is that the system of equations becomes indeterminate if one ignores (16). This reduces the number of equations by one but does not reduce the number of unknowns, because c still appears in (18) and r still appears in most equations. So the production and consumption sectors of the economy cannot be separated analytically.

4 World economy

4.1 Model

Suppose now that the world economy is divided into M (> 1) countries with identical technologies, tastes, and market structures everywhere. The countries are distinguished by a superscript $m \in \{1, \ldots, M\}$. Inputs and outputs in a given country m are denoted by lower-case letters. Upper-case letters denote world aggregates. Following Ethier [20], spillover effects are assumed to be international in scope, so that the parameters B_Y in the production function for final goods and A in the R&D technology are also the same in each country. For example, aggregate production in m is $y^m = F_Y(k_Y^m, B_Y l_Y^m, d_Y^m)$, where $B_Y = K_Y^\eta / L_Y^\varepsilon$, $K_Y = \sum_{m=1}^M k_Y^m$, and $L_Y = \sum_{m=1}^M l_Y^m$. Let a^m denote the number of intermediates invented in country m in the PV model and the number of intermediate goods markets with a quality leader from country m in the QU model. Then the number of new intermediates invented in country m in the PV model is $\dot{a}^m = F_A(k_A^m, l_A^m) A^\chi$, where $A = \sum_{m=1}^M a^m$. In the QU model, the rate of innovation in m is $i^m = F_A(k_A^m, l_A^m) A^{-(1-\chi)}$, where $A(t) = \exp[\int_{-\infty}^t I(\tau) d\tau]$ and $I = \sum_{m=1}^M i^m$. We maintain the assumption that at the outset there is only one supplier of each intermediate in the PV model and only one supplier of the highest quality producible of each intermediate in the QU case.[13] There is free trade in the final good and the intermediate goods. As a consequence, the prices of the final good (unity) and the intermediates ($p_{X_\omega}(j)$) are the same in each country. Financial capital is also freely traded internationally. So one country can finance spending on consumption, capital, or R&D by incurring debt elsewhere in the world economy, and there is a unique interest rate, r. The model covers special cases with (1) no physical capital. If the model contains physical capital, we have to make a further case distinction. The fact that the final good is traded internationally and that it can be used as investment in physical capital implies that *new* physical capital can be accumulated by imports. We must also specify, however, if it is possible to import ("old") physical capital already installed in foreign factories. Here we allow for two different cases. On the one hand, as is usual in growth theory, (2) physical capital is perfectly mobile internationally. It can be de-installed and transferred abroad, and the distribution of aggregate capital $K = \sum_{m=1}^M k_m$

[13] Tang and Wälde (2001) investigate the implications of an initial overlap of intermediate goods in a PV model. Their main result, the possible existence of a "no-growth trap" (i.e., stagnation despite the existence of a balanced-growth equilibrium with a positive rate because of unfavorable initial conditions), can be proved by showing that it holds for a closed economy and then demonstrating that the the world equilibrium replicates the integrated equilibrium. Since the equilibrium loses its symmetry properties with regard to the different intermediates, the analysis becomes much more tedious however, so we refrain from a formal exposition.

across countries can change instantaneously[14] (k^m denotes capital *used in*, not capital *owned by*, the residents of, country m). On the other hand, once installed, (3) physical capital is immobile internationally, as is usual in the static Heckscher-Ohlin theory, so that k^m is a state variable. Another case distinction relates to the issue of whether the production of intermediate goods has to be located where the respective goods have been innovated (cf. [24], Subsections 7.3 and 7.4). Again, we allow for two possibilities. On the one hand, we assume that (a) intermediates have to be produced where they have been invented. On the other hand, we allow for the existence of (b) multinational corporations or costless licensing contracts (with an enforceable commitment by the innovator not to compete with the licensee). In this latter case, innovators can reap the benefits of an innovation even though production takes place abroad. Put differently, if multinational corporations are at work or cross-border patent licensing takes place, a'^m, the number of intermediates *produced* in country m, need not equal a^m, the number of intermediates *invented* in country m. Labor is immobile internationally.

4.2 Replication of the integrated equilibrium

In this section, following the Dixit-Norman [17] approach, we investigate the conditions under which the world economy replicates the equilibrium of the hypothetical integrated economy despite the immobility of labor.[15] As in [17], this requires either the costless mobility of physical capital or a sufficiently high degree of similarity between the countries' relative factor endowments. To achieve this, we show that if the wage rate equalizes internationally, Eqs. (9)-(20) hold true in the world economy. Then we derive the conditions for the possibility of the replication of the integrated equilibrium from the requirement that the quantities produced in each country are non-negative.

Eq. (9): Producers solve the same two-stage cost minimization problem as before. Minimization of the cost of producing one unit of d_Y (stage one) leads to the same input coefficients, $a_{X_\omega(j)}$, and to the same price, p_D, of one unit of d_Y. The second-stage problem is also unchanged and leads to the same input coefficients, $a_{K_Y}(r, w, p_D, B_Y)$, $a_{L_Y}(r, w, p_D, B_Y)$, and $a_{D_Y}(r, w, p_D, B_Y)$, and to the same unit cost function, $c_Y(r, w, p_D, B_Y)$. Competitive pricing implies the validity of (9).

Eq. (10): Capital used in final goods production in country m is $k_Y^m = a_{K_Y} y^m$. Using $Y = \sum_{m=1}^M y^m$, it follows that $K_Y = \sum_{m=1}^M k_Y^m = a_{K_Y} Y$. Similarly, $L_Y = a_{L_Y} Y$. Inserting this into (2) proves (10).

Eq. (11): The intermediate goods producers' cost minimization problem is the same as before. So the input coefficients, $a_{K_{X_\omega(j)}}(r, w)$ and $a_{L_{X_\omega(j)}}(r, w)$, and the unit cost function, $c_x(r, w)$,

[14] As, for example, in Ruffin's [43] two-country Solow [47] model.

[15] We do not address the question of whether other equilibria with non-replication exist. So, to be more precise, we should say *it is possible that* the world economy replicates the equilibrium of the hypothetical integrated economy.

are unaltered. The (homothetic) demands for the intermediates by firms in the final goods sector in country m are $a_{X_{\omega(j)}}a_{D_Y}(r,w,p_D,B_Y)y^m$. As in the integrated economy, world-wide demand is $a_{X_{\omega(j)}}a_{D_Y}(r,w,p_D,B_Y)Y$. Since the unit cost function and the world-wide demand, $a_{X_{\omega(j)}}a_{D_Y}(r,w,p_D,B_Y)Y$, for intermediates as well as the market structure are the same as in the integrated economy, so is the monopoly price, p, in (11).

Eq. (12): It follows immediately that monopoly profits, π, obey (12).

Eq. (13): Analogously to Subsection 3.1, p_D can be rewritten as in (13) and the input coefficients satisfy $a_X = p_D/(A^{1-\mathcal{D}}p)$.

Eq. (14): Cost minimization in the R&D sector yields the same input coefficients, $a_{K_A}(r,w)$ and $a_{L_A}(r,w)$, and the same cost function, $c_A(r,w)$, as before. Together with constant returns to scale of F_A, it follows that $\sum_{m=1}^M F_A(k_A^m, l_A^m) = F_A(K_A, L_A)$. In the PV model, the validity of (6) thus follows from $\dot{a}^m = F_A(k_A^m, l_A^m)A^\chi$. In the QU model, $i^m = F_A(k_A^m, l_A^m)A^{-(1-\chi)}$. Hence $I = \sum_{m=1}^M i^m = F_A(K_A, L_A)A^{-(1-\chi)}$, where $A(t) = \exp[\int_{-\infty}^t I(\tau)d\tau]$. As above, differentiating A gives $\dot{A} = AI$ and (6). Obviously. the definition of the value of an innovation in (14) is valid in the PV model. In the QU model, the validity of the definition follows from the observation that the innovation rate, I, is equal to $\dot{A}/A$.

Eq. (15): Since $F(k_A^m, l_A^m) = 1$ is worth $A^{\chi-\mathcal{D}}v$ and costs $c_A(r,w)$, free entry into R&D implies (15).

Eqs. (16) and (17): The households' maximization problem remains unchanged, so (16) and (17) follow.

Eq. (18): Let s^m denote country m's net exports of the final good. The world's net exports as a whole must be zero: $\sum_{m=1}^M s^m = 0$. The supply of final goods equals demand in country m if $y^m = \dot{k}^m + n^m c + s^m$. Summing over m, using $\sum_{m=1}^M s^m = 0$ and $\sum_{m=1}^M n^m = N$, yields $Y = \dot{K} + Nc$, i.e. (18).

Eq. (19): It has already been shown that the demand for intermediates, $a_X a_{D_Y}(r,w,p_D,B_Y)Y$, and the input coefficients, $a_X = p_D/p$, are the same as in the integrated economy. So (19) continues to give the condition for an equilibrium in the markets for intermediate goods.

Eqs. (20): The conditions for equality of the supply and demand in the markets for physical capital and labor in country m can be written as:

$$\begin{pmatrix} k^m \\ n^m l \end{pmatrix} = \begin{pmatrix} a_{K_Y}(r,w,p_D,B_Y) & a_{K_X}(r,w) & a_{K_A}(r,w) \\ a_{L_Y}(r,w,p_D,B_Y) & a_{L_X}(r,w) & a_{L_A}(r,w) \end{pmatrix} \begin{pmatrix} y^m \\ a'^m X \\ A^{1-\chi}\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}} (i^m)^{\mathcal{D}} \end{pmatrix} + \begin{pmatrix} a_{k_l}(r,w) \\ a_{l_l}(r,w) + a_{l_l'}(r,w) \end{pmatrix} n^m \dot{l}. \tag{21}$$

As in (20), the four terms on the right-hand sides of these equations are capital and labor in their four different uses in country m. For instance, capital in R&D is $k_A^m = a_{K_A}(r,w)\dot{a}^m/A^\chi$ in the PV model (i.e. for $\mathcal{D}=0$) and $k_A^m = a_{K_A}(r,w)A^{1-\chi}i^m$ in the QU model. This follows from the R&D technologies $\dot{a}^m = F(k_A^m, l_A^m)A^\chi$ and $i^m = F_A(k_A^m, l_A^m)A^{-(1-\chi)}$, respectively. Adding these equations for all M countries (using the fact that $\sum_{m=1}^M a'^m$ equals A in the PV model and unity in the QU model) proves the validity of (20).

Eq. (7): As for the additional two equations for the QU model, the former follows by definition, while $I = \dot{A}/A$ has already been derived in the proof of the validity of (14).

This proves that there exist quantities y^m, a'^m, and $(\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}}$ such that the equilibrium conditions of the integrated economy are fulfilled. It remains for us to find out the conditions under which these quantities are *non-negative*. In doing so, we have to make the two case distinctions mentioned above: ("old") physical capital, if not (1) absent from the model, is (2) mobile or (3) immobile, and intermediates (a) need or (b) need not be produced where they have been invented. Some additional notation is helpful. For the sake of convenience, here and in what follows we drop the arguments of the input coefficients functions. (a) In the absence of multinational corporations or cross-border patent licensing ($a'^m = a^m$), let

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} \equiv \begin{pmatrix} k^m \\ n^m l \end{pmatrix} - \begin{pmatrix} a_{K_X} & a_{k_l} \\ a_{L_X} & a_{l_l} + a_{l'_l} \end{pmatrix} \begin{pmatrix} a'^m X \\ n^m \dot{l} \end{pmatrix}.$$

With this definition, (21) can be rewritten as

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} a_{K_Y} & a_{K_A} \\ a_{L_Y} & a_{L_A} \end{pmatrix} \begin{pmatrix} y^m \\ A^{1-\chi}\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix}. \tag{22}$$

(b) Similarly, with multinational corporations or cross-border patent licensing, let

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} \equiv \begin{pmatrix} k^m \\ n^m l \end{pmatrix} - \begin{pmatrix} a_{k_l} \\ a_{l_l} + a_{l'_l} \end{pmatrix} n^m \dot{l},$$

so that (21) becomes

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} a_{K_Y} & a_{K_X} & a_{K_A} \\ a_{L_Y} & a_{L_X} & a_{L_A} \end{pmatrix} \begin{pmatrix} y^m \\ a'^m X \\ A^{1-\chi}\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix}. \tag{23}$$

The vector $(\bar{k}^m, \bar{l}^m)'$ gives the factor supplies *net of the quantities used in the activities which cannot be moved internationally*, i.e., human capital accumulation and, in the absence of multinational corporations or international patent licensing, intermediate goods production. The following theorem states the necessary and sufficient conditions for (22) and (23) to have non-negative solutions and, hence, for the feasibility of the replication of the integrated equilibrium:

Theorem 1: *Suppose an integrated equilibrium with $Y(t) \geq 0$, $\dot{A}(t)^{1-\mathcal{D}}I(t)^{\mathcal{D}} \geq 0$, and $\dot{l}(t) \geq 0$ for all t exists. If physical capital is (1) absent from the model or (2) mobile, then the replication of the integrated equilibrium is an equilibrium of the world economy if and only if for all t and for all $m \in \{1, \ldots, M\}$, $\bar{l}^m(t) \geq 0$. If (b) $a^m(t)$ need not equal $a'^m(t)$ ($m \in \{1, \ldots, M\}$), this condition is automatically satisfied. (3) If physical capital is immobile, then the replication of the integrated equilibrium is an equilibrium of the world economy if and only if for all t and for all $m \in \{1, \ldots, M\}$, the following two conditions are satisfied. First, $(\bar{k}^m(t), \bar{l}^m(t))' \geq \mathbf{0}$. Second, when $a'^m = a^m$,*

$$\exists \begin{pmatrix} y^m \\ (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix} \geq \mathbf{0} : \sum_{m=1}^{M} \begin{pmatrix} y^m \\ (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} Y \\ \dot{A}^{1-\mathcal{D}}I^{\mathcal{D}} \end{pmatrix} \text{ and (22) holds,}$$

and when a'^m need not equal a^m,

$$\exists \begin{pmatrix} y^m \\ a'^m \\ (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix} \geq \mathbf{0} : \sum_{m=1}^{M} \begin{pmatrix} y^m \\ a'^m \\ (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} Y \\ A^{1-\mathcal{D}} \\ \dot{A}^{1-\mathcal{D}}I^{\mathcal{D}} \end{pmatrix} \text{ and (23) holds.}$$

Proof: (a) Suppose a'^m must equal a^m, so that (22) applies. The right-hand side of the second equality in (22) equals zero for $(y^m, (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}})' = \mathbf{0}$ and increases as y^m or $(\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}}$ rises. So a non-negative solution $(y^m, (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}})'$ exists if and only if the left-hand side is non-negative ($\bar{l}^m \geq 0$). This proves the possibility of replicating the integrated equilibrium if (1) the model does not contain physical capital, because the first equality in (22) drops out in this case. If (2) physical capital is present and mobile, k^m is endogenous and must be non-negative. As the capital inputs in the integrated equilibrium are non-negative, inserting a non-negative solution $(y^m, (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}})'$ to the second equality in (22) into the first equality yields a non-negative equilibrium value for k^m. (b) With multinational corporations or international patent licensing, (23) becomes relevant. If capital is (1) absent or (2) mobile, then by the same reasoning as in case (a), a non-negative solution $(y^m, a'^m, (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}})'$ to (23) exists if and only if $\bar{l}^m \geq 0$. Here, $\bar{l}^m \geq 0$ is implied by

$$\bar{l}^m = \frac{n^m}{N}\left[lN - (a_{l_l} + a_{l'_l})N\dot{l}\right],$$

the fact that the term in square brackets equals employment in final goods production and R&D in the integrated equilibrium, and the supposition that Y and $\dot{A}^{1-\mathcal{D}}I^{\mathcal{D}}$ are non-negative in the integrated equilibrium. For the case of (3) immobile physical capital, the theorem merely restates formally the condition that a non-negative solution to (22) or (23) exists, depending on whether a'^m need or need not equal a^m. *q.e.d.*

By providing the necessary and sufficient conditions for the world economy to replicate the equilibrium of the hypothetical integrated economy, Theorem 1 extends the classical Dixit-Norman approach to international trade to a broad class of growth models. The mobility (or absence) of capital and the similarity of relative factor endowments make replication more likely. The presence of multinational corporations or international patent licensing is a sufficient condition in the former case. In the latter case, it enhances the likelihood of replication. This is because y^m and $(\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}}$ values which solve (22) plus $a'^m = a^m$ solve (23), but the y^m and $(\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}}$ values from a solution $(y^m, a'^m, (\dot{a}^m)^{1-\mathcal{D}}(i^m)^{\mathcal{D}})'$ to (23) with $a'^m \neq a^m$ do not solve (22). Notice that we have not imposed a balanced-growth assumption. The analysis applies to the models' transitional dynamics as well.

4.3 Digression: Two-country example

The usual graphical illustration of the two-country special case of the Dixit-Norman approach works here as well and provides an intuition for Theorem 1. Consider the two-country example ($M = 2$). Using the fact that capital and labor in R&D equal $K_A = a_{K_A}\dot{A}/A^{\chi}$ and $L_A = a_{L_A}\dot{A}/A^{\chi}$, respectively, in the PV model and $K_A = a_{K_A}IA^{1-\chi}$ and $L_A = a_{L_A}IA^{1-\chi}$, respectively, in the QU model, (22) and (23) can be rewritten as

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} K_Y & K_A \\ L_Y & L_A \end{pmatrix} \begin{pmatrix} \frac{y^m}{Y} \\ \left(\frac{\dot{a}^m}{\dot{A}}\right)^{1-\mathcal{D}} \left(\frac{i^m}{I}\right)^{\mathcal{D}} \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} K_Y & A^{1-\mathcal{D}}K_X & K_A \\ L_Y & A^{1-\mathcal{D}}L_X & L_A \end{pmatrix} \begin{pmatrix} \frac{y^m}{Y} \\ \frac{a'^m}{A} \\ \left(\frac{\dot{a}^m}{\dot{A}}\right)^{1-\mathcal{D}} \left(\frac{i^m}{I}\right)^{\mathcal{D}} \end{pmatrix}$$

for $m \in \{1, 2\}$. Thus, the replication of the integrated-equilibrium is feasible if it is possible to assign non-negative portions of the integrated equilibrium production vectors of the internationally mobile activities to the two countries. Figures 1 and 2 offer a graphical illustration. The width of the boxes in the figures is $\bar{l}^1 + \bar{l}^2$ and the height is $\bar{k}^1 + \bar{k}^2$, where it is assumed that $(\bar{k}^m, \bar{l}^m)' \geq \mathbf{0}$ for $m \in \{1, 2\}$.[16] (3) With immobile physical capital, both $\bar{k}^1$ and $\bar{k}^2$ are given. (2) With mobile physical capital, k^1 and k^2 are not fixed, but their sum $\bar{k}^1 + \bar{k}^2$ is. This is because $\bar{k}^1 + \bar{k}^2$ equals total capital K (a) minus $(A^{1-\mathcal{D}}K_X + Nk_l)$ or (b) minus Nk_l, where A, K_X, and k_l are the integrated-equilibrium quantities.

[16] Of course, the boxes grow as capital and labor grow.

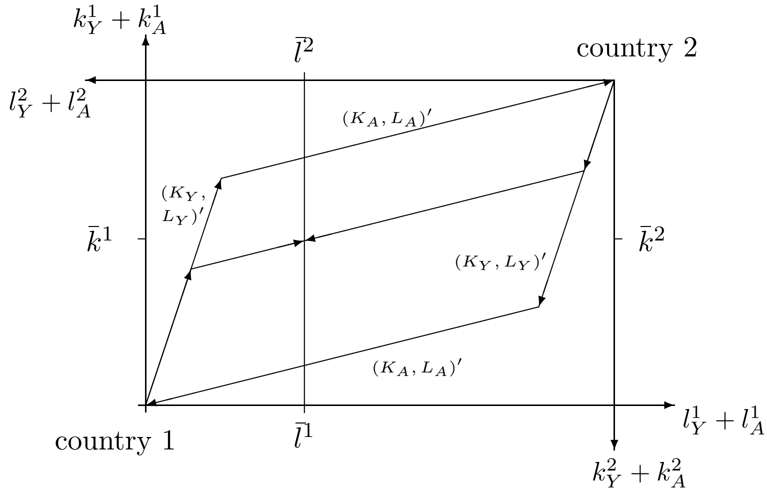

Figure 1: Two countries without multinational firms or international patent licensing

(1) In the absence of physical capital, the boxes degenerate to horizontal lines. For the time being, we ignore this case.

Figure 1 applies to case (a) with $a'^m = a^m$. The figure shows the integrated-equilibrium production vectors $(K_Y, L_Y)'$ and $(K_A, L_A)'$.[17] The replication of the integrated equilibrium is feasible if it is possible to assign non-negative portions of these vectors to the two countries. Suppose (3) physical capital is immobile. Then the endowment point $(\bar{k}^1, \bar{l}^1)$ (or $(\bar{k}^2, \bar{l}^2)$ from country 2's perspective) is determinate. The economy replicates the integrated economy if and only if the endowment point is located in the parallelogram formed by the integrated-equilibrium production vectors. If so, the input vectors $(k_Y^1, l_Y^1)'$, $(k_A^1, l_A^1)'$, $(k_Y^2, l_Y^2)'$, and $(k_A^2, l_A^2)'$ are uniquely determined. (2) Next consider the case of capital mobility. Since the sum $\bar{k}^1 + \bar{k}^2$ is determinate but not its elements, all values $k_Y^1 + k_A^1$ and $k_Y^2 + k_A^2$ on that portion of the vertical line through $\bar{l}^1$ and $\bar{l}^2$ which is located inside the parallelogram are consistent with the replication of the integrated-equilibrium. (1) In the absence of physical capital, the box in Figure 1 degenerates to a line. $\bar{l}^1$ and $\bar{l}^2$ determine the endowment point on this line, and any division of the integrated equilibrium production vectors between countries such that labor is fully employed in both countries implies that the integrated equilibrium is being replicated. If more than one production vector entails a strictly positive labor input, the division is indeterminate.

Figure 2 applies to case (b) with a'^m endogenous. It depicts the integrated-equilibrium production vectors $(K_Y, L_Y)'$, $A^{1-\mathcal{D}}(K_X, L_X)'$, and $(K_A, L_A)'$. Replication is feasible if it is possible to assign non-negative portions of these three integrated-equilibrium production vectors to the two countries.

[17] The ranking of the different productive activities in terms of capital intensity is inessential.

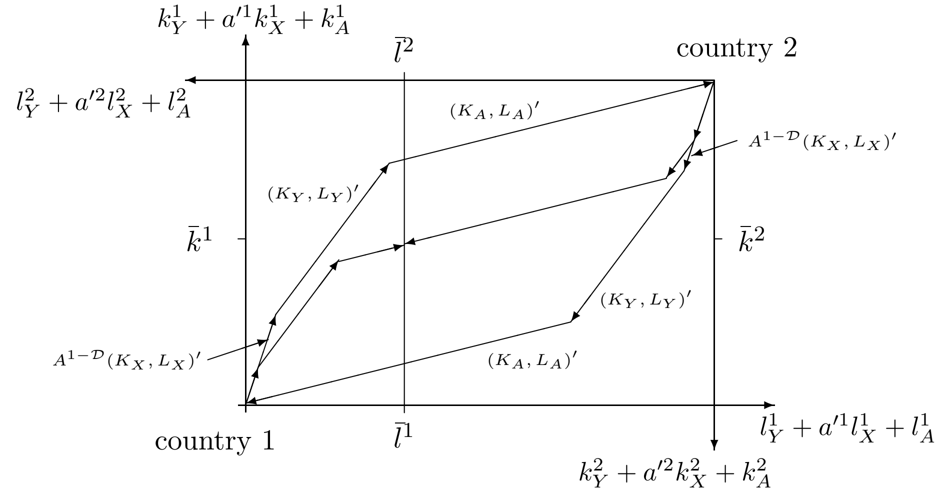

Figure 2: Two countries with multinational firms or international patent licensing

(3) With immobile physical capital, the economy replicates the integrated economy if and only if the determinate endowment point is located in the hexagon formed by the three integrated-equilibrium production vectors. Even so, the input vectors $(k_Y^1, l_Y^1)'$, $(a'^1 k_X^1, a'^1 l_X^1)'$, $(k_A^1, l_A^1)'$, $(k_Y^2, l_Y^2)'$, $(a'^2 k_X^2, a'^2 l_X^2)'$, and $(k_A^2, l_A^2)'$ are not uniquely determined, as is usual when the number of factors is less than the number of their different uses. (2) If capital is mobile internationally, all values $k_Y^1 + k_A^1$ and $k_Y^2 + k_A^2$ on that portion of the vertical line through $\bar{l}^1$ and $\bar{l}^2$ which is located inside the hexagon are consistent with the replication of the integrated-equilibrium, so there is another degree of freedom in splitting up world production vectors between countries. Whenever at least one component of the integrated equilibrium production vector $A^{1-\mathcal{D}}(K_X, L_X)'$ is strictly positive, the hexagon formed by the three production vectors $A^{1-\mathcal{D}}(K_X, L_X)'$, $(K_Y, L_Y)'$, and $(K_A, L_A)'$ in Figure 2 is larger than the parallelogram formed by the two production vectors $(K_Y, L_Y)'$ and $(K_A, L_A)'$ in Figure 1. This illustrates the fact that multinationals or patent licensing make the replication of the integrated equilibrium more likely.[18] (1) In the absence of physical capital, the analysis for the case $a'^m = a^m$ applies.

4.4 Implications

Besides generalizing the Dixit-Norman approach to a broad class of growth models, the analysis has two important implications. First, and most importantly, our Dixit-Norman approach provides a

[18]With more than one consumption good, as in [24] (Chapter 7), there would be more integrated-equilibrium production vectors in Figures 1 and 2. The possibility of the replication of the integrated equilibrium would become more likely. If the conditions for replication were satisfied, there would be another degree of freedom in splitting up the world production levels between countries. This would render the equilibrium in case (a) indeterminate.

fruitful way of systematically investigating transitional dynamics in open-economy growth models. The analysis of the transitional dynamics of open-economy growth models is a dauntingly difficult task.[19] Theorem 1 offers a way of reducing the complexity drastically. Rather than directly addressing the the problem, one can first analyze the transitional dynamics of the hypothetical integrated economy and then check if the conditions of the theorem are satisfied. If so, we can conclude that world aggregates behave as they would in the integrated economy, factor prices equalize, and certain fractions of world-wide economic activity are conducted in the separate countries. This is a direct corollary to Theorem 1:

Theorem 2: *Suppose the conditions of Theorem 1 are satisfied. Then the transitional dynamics of the world economy replicate the transitional dynamics of the integrated economy, and (22) or (23) determines the national output levels.*

In the next section, we show how this simple idea allows us to go a long way towards understanding the transitional dynamics of the open-economy versions of several prominent special cases of our general model. Second, the analysis sheds light on the question of whether long-run growth is faster in the presence of international trade and international knowledge spillovers than in their absence. To see this, consider a country which is in complete economic isolation (without international trade and without international spillovers). Then international economic integration makes the country a part of a larger world economy with international trade and international spillovers.[20] Another direct corollary to Theorem 1 provides the answer to the question of whether the country will grow faster in the long run:

Theorem 3: *Suppose the conditions of Theorem 1 are satisfied. Then international economic integration raises the economy's long-run growth rate if and only if scale effects prevail in the integrated equilibrium.*

We return to this question in Subsection 5.8.

5 Examples

In subsections 5.2-5.6 of this section, we show that our general model encompasses, as special cases, important models such as those of Solow [47], Romer [41], Lucas [35], Romer [42], Grossman and

[19] See, for example, [24] (Chapter 8), [25], [15], [54], [19], [50], [4], and [6]. It appears fair to say that a *systematic* investigation of the transitional dynamics of endogenous growth models does not exist.

[20] As noted in Section 1, the impact of trade if knowledge flows freely is already discussed in [24] (Chapters 8 and 9) and [40].

Helpman [24] (Chapters 3 and 4), Jones [27], and Segerstrom [45]. Ignoring open economy issues for the moment, we state the growth rate of aggregate output, $g_Y \equiv \dot{Y}/Y$, in a balanced-growth equilibrium of the integrated economy (a supplement to this paper provides a "toolkit" for calculating these growth rates) and the dynamic properties of those models whose transitional dynamics have been investigated. Equipped with these results, we return to the open economy. In Subsection 5.7, we present original results on the transitional dynamics of the models introduced in the preceding subsections. In Subsection 5.8, we check under which conditions international economic integration leads to faster growth in a balanced-growth equilibrium.

5.1 Balanced growth

A balanced-growth equilibrium is characterized by four requirements. First, growth rates are constant, the state variables' growth rates are non-negative (because of the absence of depreciation), and consumption growth is such that the transversality conditions for the consumers' maximization problem are satisfied. Second, from (18), $Y/K = \dot{K}/K + Nc/K$. So we assume that $\dot{Y}/Y = \dot{K}/K = \dot{c}/c + g_N$ in a balanced-growth equilibrium. Third, the growth rates of capital and labor in their different respective uses are identical. Fourth, the proportion of labor income to Y is constant. For the sake of convenience, we denote aggregate labor supply as $L \equiv Nl$. Further, let $\gamma \equiv (1-\alpha)/\alpha$ in the PV model and $\gamma \equiv \log \lambda$ in the QU model, and denote the share of factor z in the unit cost of producing Z as θ_{z_Z}. For instance, labor's share in the cost of producing final goods is $\theta_{L_Y}(r, w, p_D, B_Y) \equiv wa_{L_Y}(r, w, p_D, B_Y)$. There are basically two ways of simplifying our general model. For one thing, it can be assumed that human capital accumulation and/or R&D are not actively performed because $F_l(k_l, l_l, l'_L)$ and/or $F_A(K_A, L_A)$ are identically zero. This gives rise to a distinction between models without human capital accumulation and without R&D (Subsection 5.2), models with human capital accumulation and without R&D (Subsection 5.3), models with R&D and without human capital accumulation (where the subcases of diminishing and constant returns in R&D are analyzed separately in Subsections 5.4 and 5.5, respectively), and models with R&D and with human capital accumulation (Subsection 5.6). For another, by assuming constant (or zero) production elasticities, the cost shares become constants (zero). For instance, if labor L_Y affects final goods production only through the term $L_Y^{1-\alpha}$, then $\theta_{L_Y} = 1 - \alpha$.

5.2 Models without human capital accumulation and without R&D

Suppose $F_l(k_l, l_l, l'_L)$ is identically zero, so that $g_l = 0$. Suppose also $F_A(K_A, L_A)$ is identically zero, such that R&D is not performed either. For the sake of simplicity, we delete argument D_Y from the production function, which becomes $Y = F_Y(K_Y, B_Y L_Y)$. We consider four prominent special cases which correspond to the first four admissible specifications for η and ε mentioned in Section 2.

Example 1 (Solow [47]): $\eta = \varepsilon = 0$. Then:

$$g_Y = g_N.$$

Productivity growth, $g_Y - g_N$, is zero. There is a unique trajectory converging to the balanced-growth equilibrium.

Example 2 (Arrow [7], Sheshinski [46]): $0 < \eta < 1$, $\varepsilon = 0$. Then:

$$g_Y = \frac{1}{1-\eta} g_N.$$

If the population grows ($g_N > 0$), so does labor productivity: $g_Y - g_N = \eta g_N/(1-\eta) > 0$. There is a unique trajectory converging to the balanced-growth equilibrium.

Example 3 (Romer [41]): $\eta = 1$, $\varepsilon = 0$, $g_N = 0$. Letting $D_z f$ denote the partial derivative of a function f with respect to its z-th argument, we get:

$$g_Y = \frac{D_1 F_Y(1, L) - \rho}{\sigma}.$$

In contrast to Examples 1 and 2, scale effects prevail. The model has no transitional dynamics. It enters its balanced-growth path immediately.

Example 4: $\eta = \varepsilon = 1$, g_N. Here we have:

$$g_Y = \frac{D_1 F_Y(1, 1) - \rho}{\sigma}.$$

Like the previous one, this model has no transitional dynamics.

5.3 Models with human capital accumulation and without R&D

Suppose $F_A(K_A, L_A)$ is identically zero. That is, R&D is not performed. As in the previous subsection, we can delete D_Y in the production function.

Example 5 (Uzawa [53]): Uzawa makes the additional assumptions that learning-by-doing effects are absent and that the students' labor is the only input in the education process: $\eta = \varepsilon = 0$, $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$. One obtains:

$$g_Y = g_N + \frac{\frac{1}{a_{l_l}} - \rho}{\sigma}.$$

There is a unique trajectory converging to the balanced-growth equilibrium (see [13], Theorem 1, p. 1056, [9], Subsection 5.2.2, [21], Section 3, [1], Sections 4 and 5).

Example 6 (Lucas [35]): For the sake of simplicity, ignore population growth here and set $N = 1$ such that L_Y is human capital per capita in production. Assume that positive externalities emanate from L_Y. In terms of (2), $\eta = 0$ such that $B_Y = L_Y^{-\varepsilon}$ with $\varepsilon < 0$. All other assumptions of the Uzawa model are maintained. Then:

$$g_Y = \frac{\frac{1}{a_{l_l}} - \rho}{\sigma + \frac{\varepsilon}{1-\varepsilon}}.$$

Benhabib and Perli [10] analyze the model's transitional dynamics for the Cobb-Douglas special case. Suppose $1/a_{l_l} > \rho$ (otherwise increases in the input coefficient, a_{l_l}, and the discount rate, ρ, raise g_Y) and $\sigma > 1 - \rho a_{l_l}(1 - \theta_{K_Y})/(1 - \theta_{K_Y} - \varepsilon)$. Then there is (locally) a unique trajectory converging to the balanced-growth equilibrium (see Proposition 1 in [10], p. 123, with $\beta = \theta_{K_Y}$, $\delta = 1/a_{l_l}$, and $\gamma = -\varepsilon$).[21]

Example 7 (based on Mulligan and Sala-i-Martin [38], and others): As in the Uzawa model, there are no externalities ($\eta = \varepsilon = 0$). Capital and other people's labor are now used in human capital formation. The interest rate, r, and r/w are determined by

$$r = \frac{1}{c_l\left(\frac{r}{w}, 1\right)}.$$

and

$$r = D_1 F_Y\left[\frac{a_{K_Y}\left(\frac{r}{w}, 1\right)}{a_{L_Y}\left(\frac{r}{w}, 1\right)}, 1\right].$$

g_Y is obtained by inserting the solution for r into the Ramsey rule (16). Stability analyses ignore input l_l' in the education technology. There is, then, a unique trajectory converging to the balanced-growth equilibrium (see [38], Interesting Result 6, p. 759, [12], Proposition 2, p. 160, [37], Subsection 3.3, [31], Subsection 3.1).

5.4 Models with diminishing returns in R&D and without human capital accumulation

Suppose the returns to existing knowledge in R&D are diminishing: $\chi < 1$. Human capital cannot be accumulated because $F_l(k_l, l_l, l_l')$ is identically zero. Here and in the subsequent analysis, we maintain the assumption of no externalities in final goods production: $\varepsilon = \eta = 0$.

Example 8 (Segerstrom [45]): Segerstrom analyzes a QU model with diminishing returns in R&D and without capital. Labor, which is, thus, the only input in the production of intermediates and in R&D, is not used in final goods production. In a balanced-growth equilibrium, $\dot{A}/A = g_N/(1-\chi)$ and

$$g_Y = \left(1 + \frac{\gamma}{1-\chi}\right) g_N.$$

[21] Benhabib and Perli [10] show that, for other parameter values, the equilibrium may be indeterminate. Another route often taken to arrive at indeterminacy results in human capital growth models is to endogenize the supply of labor.

The analysis applies to the PV case as well. There is a unique trajectory converging to the balanced-growth equilibrium (see [45], p. 1300).

Example 9 (Jones [27]): Jones considers a PV model in which final goods are produced using labor and intermediates according to $Y = L_Y^{1-\alpha} D_Y^{\alpha}$. Physical capital is the only input in the production of the intermediates, and R&D does not require capital. Then:

$$g_Y = \left(1 + \frac{\alpha}{1-\alpha}\frac{\gamma}{1-\chi}\right) g_N.$$

In the PV model considered by Jones, $\gamma \equiv (1-\alpha)/\alpha$ and, hence, $g_Y = [1 + 1/(1-\chi)]g_N$. The analysis applies to the QU model as well. Steger [48] analyzes the transitional dynamics of calibrated versions of the Jones model. In the numerical specifications he considers, there is (locally) a unique trajectory converging to the balanced-growth equilibrium ([48], p. 19).

Example 10: More generally, assume the production functions F_Y, F_X, and F_A are Cobb-Douglas, so that the cost shares θ_{zZ} are exogenous. Then:

$$g_Y = \left[1 + \frac{\theta_{D_Y}\frac{\gamma}{1-\chi}}{1 - \theta_{K_Y} - \theta_{D_Y}\left(\theta_{K_X} + \theta_{K_A}\frac{\gamma}{1-\chi}\right)}\right] g_N.$$

With regard to productivity growth, the models with diminishing returns in R&D and without human capital accumulation are quite similar to the Arrow-Sheshinski model (Example 2): in both cases, population growth is the necessary and sufficient condition for sustained productivity growth.

5.5 Models with constant returns in R&D and without human capital accumulation

With non-diminishing returns to knowledge in R&D ($\chi = 1$), we have $\dot{A}/A = F_A(K_A, L_A)$ in the PV model and $I = F_A(K_A, L_A)$ in the QU model. In order for $\dot{A}/A$ or I to be bounded, capital must not be an argument of the R&D production function and the supply of labor must not rise due to human capital accumulation nor because of population growth. So $F_A(K_A, L_A) = L_A/a_{L_A}$, where a_{L_A} is exogenous, $F_l(k_l, l_l, l_l')$ is identically zero (so $g_l = 0$), and $g_N = 0$. This class of growth models contains two of the most prominent ones.

Example 11 (Grossman and Helpman [24], Chapters 3 and 4): As in Example 8, the model does not contain capital. Labor is the only input in the production of intermediates and in R&D, but is not used in final goods production. Grossman and Helpman consider both the PV version ([24], Chapter 3) and the QU version ([24], Chapter 4) of the model. In a balanced-growth equilibrium:

$$g_Y = \frac{(\mu - 1)\frac{L}{a_{L_A}} - \rho}{\sigma - 1 + \frac{\mu}{\lambda}}.$$

Grossman and Helpman [24] assume logarithmic utility ($\sigma = 1$). Thus, the growth rate is $g_Y = (1-\alpha)[(1-\alpha)L/a_{L_A} - \alpha\rho]/\alpha$ in the PV case and $g_Y = \log\lambda[(\lambda-1)L/a_{L_A} - \rho]/\lambda$ in the QU case. The economy jumps on its balanced-growth path (see [24], p. 61, for the PV model and [24], p. 96, for the QU model).

Example 12 (Romer [42]): Romer considers a PV model with similar assumptions as in Example 9. Final goods production obeys $Y = L_Y^{1-\alpha} D_Y^{\alpha}$, and physical capital is the only input in the production of the intermediates. Then:

$$g_Y = \frac{\left(1-\frac{1}{\mu}\right)\frac{\alpha}{1-\alpha}\frac{L}{a_{L_A}} - \rho}{\sigma - 1 + \frac{1-\alpha}{\alpha\gamma} + \left(1-\frac{1}{\mu}\right)\frac{1}{\gamma}}.$$

As Romer [42] considers the PV variant of the model (with $\mu = 1/\alpha$ and $\gamma \equiv (1-\alpha)/\alpha$), this boils down to $g_Y = (\alpha L/a_{L_A} - \rho)/(\sigma+\alpha)$. The analysis applies to the QU model as well. Arnold [3] (Theorem 1, p. 74) proves that there is (locally) a unique trajectory converging to the balanced-growth equilibrium.[22]

Example 13: More generally, assume the production functions F_Y and F_X are Cobb-Douglas, so that the cost shares θ_{z_Z} are exogenous. Then:

$$g_Y = \frac{\left(1-\frac{1}{\mu}\right)\frac{\frac{L}{a_{L_A}}}{\frac{\theta_{L_Y}}{1-\theta_{K_Y}-\theta_{L_Y}} + \frac{\theta_{L_X}}{\mu}} - \rho}{\sigma - 1 + \frac{1-\theta_{K_Y}-\theta_{D_Y}\theta_{K_X}}{\gamma\theta_{D_Y}} + \left(1-\frac{1}{\mu}\right)\frac{\frac{1-\theta_{K_Y}-\theta_{D_Y}\theta_{K_X}}{\gamma\theta_{D_Y}}}{\frac{\theta_{L_Y}}{1-\theta_{K_Y}-\theta_{L_Y}} + \frac{\theta_{L_X}}{\mu}}}.$$

5.6 Models with diminishing returns in R&D and with human capital accumulation

This class of models is characterized by $\chi < 1$ and $g_l > 0$.

Example 14 (Arnold [2], [5]): Following Examples 8 and 11, the model does not contain capital, the final good is produced from the intermediates alone, and labor is the only input in the production of intermediates and in R&D. Following Examples 5 and 6, the students' labor is the only input in the education process: $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$. For the sake of simplicity, let $g_N = 0$. The PV version of this model is analyzed in Arnold [2] and the QU version in Arnold [5]. In a balanced-growth equilibrium:

$$g_Y = \frac{\frac{1}{a_{l_l}} - \rho}{\sigma - 1 + \frac{1}{1+\frac{\gamma}{1-\chi}}}.$$

There is a unique trajectory converging to the balanced-growth equilibrium (see [2], Propositions 2, p. 91, and 9, p. 103).

[22] Benhabib, Perli, and Xie [11] show that indeterminacy may arise with a more general production technology for final goods, which allows for complementarity between the intermediates.

Example 15: More generally, suppose the production functions F_Y, F_X, and F_A are Cobb-Douglas. Under the maintained assumption that $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$, one obtains:

$$g_Y = \frac{\frac{1}{a_{l_l}} - \rho}{\sigma - 1 + \frac{1-\theta_{K_Y}-\theta_{D_Y}\theta_{K_X}-\theta_{D_Y}\frac{\gamma}{1-\chi}\theta_{L_A}}{\theta_{D_Y}\left(\theta_{L_X}+\frac{\gamma}{1-\chi}\theta_{L_A}\right)}}.$$

5.7 Transitional dynamics

The *balanced-growth equilibria* of the open-economy versions of many of the models introduced in Subsections 5.2-5.6 have been analyzed in the literature: for instance, the role of international flows of physical capital in raising GNP in the open-economy Solow [47] model are well understood, Grossman and Helpman [24] provide open-economy versions for their models, Rivera-Batiz and Romer [40] consider a symmetric two-country version of the Romer [42] model, and the analyis in Arnold [2] includes the open economy.[23] Little is known, however, about the *transitional dynamics* of many of these open-economy growth models. The dynamics of the open-economy Solow model are well understood. But, to the best of our knowledge, the two-country PV version of Example 11 is the only open-economy R&D model (Examples 8-15) whose transitional dynamics are known (see Grossman and Helpman [24], Section 8.1, and Wälde [54]). A systematic account of the transitional dynamics of the open-economy versions of the learning-by-doing (Examples 2-4) and human capital models (Examples 5-7) does not exist either. In this subsection, we use Theorem 2 to to fill this gap in the literature and to provide a systematic restatement of the existing stability results. Theorem 2 implies that if the conditions of Theorem 1 are satisfied, the transitional dynamics of the world economy are identical to the transitional dynamics of the hypothetical integrated economy, with a certain fraction of world-wide economic activity located in each country.[24] Notice that for all the models introduced except for the generalizations of the three categories of R&D growth models (Examples 10, 13, and 15), stability results exist (and have been cited in Subsections 5.2-5.6). For Examples 1-5, 7, 8, 11, and 14, global results exist; for Examples 6, 9, and 12, the results apply to local dynamics only. Additional simplifying assumptions are made in Examples 6 (Cobb-Douglas production function in the final goods sector, $1/a_{l_l} > \rho$), 7 (l_l' dropped in the education technology), and 9 (numerical analysis only). Notice further that in all cases, there is a unique trajectory converging to the balanced-growth equilibrium. Examples 1, 2, 5-9, 12, and 14 display transitional dynamics, while in Examples 3, 4, and

[23] Dinopoulos and Segerstrom [16] analyze the two-country version of the Segerstrom [45] model with endogenous human capital accumulation à la Findlay-Kierzkowski [22].

[24] An immediate implication is that if the integrated economy does not feature cycles or indeterminacy, the world economy does not either.

11, convergence takes place instantaneously. Equipped with Theorem 2 and with this observation, we can now discuss the transitional dynamics of the open-economy versions of Examples 1-9, 11, 12, and 14.

Examples 1-7: In models without R&D, the distinction between cases (a) and (b) is immaterial, Eqs. (22) and (23) coincide:

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} a_{K_Y} \\ a_{L_Y} \end{pmatrix} y^m$$

for $m \in \{1, \ldots, M\}$. These equations constitute an over-determinate system. Unless all countries happen to be endowed with the same amount of physical capital per worker initially, no solution y^m exists. From Theorem 1, the world economy (3) with immobile physical capital thus does not replicate the integrated economy. (2) With mobile physical capital, the replication of the integrated equilibrium is feasible if $\bar{l}^m \geq 0$. In Examples 1-4, $\bar{l}^m = n^m l \geq 0$ is trivially true. In Examples 5-7, $\bar{l}^m = n^m[l - (a_{l_l} + a_{l_l'})\dot{l}] \geq 0$ follows from the fact that a positive amount of labor is used in production in the integrated equilibrium ($N[l - (a_{l_l} + a_{l_l'})\dot{l}] > 0$). The world equilibrium with capital mobility in Examples 1-7 can thus be described as follows. Physical capital flows equalize the interest rate and wages instantaneously. Consumers face the same budget constraints as in the integrated economy, with part of their interest income coming from abroad. Consumers choose the same consumption patterns and human capital investments as in the integrated equilibrium. The world economy as a whole approaches the integrated-economy balanced-growth path.

In terms of Figure 1 or 2, there is only one integrated economy production vector, $(K_Y, L_Y)'$. As the endowment point of a two-country world economy will generally not be located on this production vector, the mobility of physical capital is required for the replicability of the integrated equilibrium.

Examples 8 and 11:[25] Since (1) physical capital is absent, the conditions of Theorem 1 are satisfied if either (a) $a'^m = a^m$ and $\bar{l}^m \equiv n^m l - a_{L_X} X a'^m \geq 0$ (i.e., the number of intermediates produced in country m is not too large relative to country m's population size n^m) or (b) a'^m is allowed to differ from a^m. We suppose in what follows that one of these conditions is satisfied. In the Grossman-Helpman model (Example 8), the integrated economy does not have transitional dynamics. So, according to Theorem 2, the world economy jumps onto the integrated-economy balanced-growth path. In Example

[25] The approach taken here is also applicable to the two-country models considered by Grossman and Helpman [24] (Chapter 7), which are not special cases of our model because they contain two consumption goods. The only state variables in their models are a^1 and a^2. Since the only state variable in the integrated economy is, thus, $A^{1-\mathcal{D}}$, the world economy enters its balanced-growth equilibrium immediately. In the sequel, a^1 and a^2 adjust until a^1/a^2 attains its balanced-growth level.

11, the economy converges towards its balanced-growth path. From (21), we have

$$\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} = \frac{n^m l - a_{L_X} X a'^m}{a_{L_A} A^{1-\chi}}. \tag{24}$$

(a) Let $a'^m = a^m$. Then, $\dot{A}/A = L_A/(a_{L_A}A^{1-\chi})$, $a_{L_X}AX = L_X$, and $L_A + L_X = Nl = L$ in the PV version and $\dot{a}^m = i^m - Ia^m$, $Ia_{L_A}A^{1-\chi} = L_A$, $a_{L_X}X = L_X$, and $L_A + L_X = Nl = L$ in the QU variant. It follows from (24) that

$$\frac{d}{dt}\left(\frac{a^m}{A^{1-\mathcal{D}}}\right) = \frac{L}{a_{L_A}A^{1-\chi}}\left(\frac{n^m}{N} - \frac{a^m}{A^{1-\mathcal{D}}}\right).$$

The fraction in front of the brackets on the right-hand side is a constant in Example 11 and, since $\dot{A}/A$ converges to $g_N/(1-\chi)$, it converges to a constant in Example 8.[26] These observations enable us to describe the models' transitional dynamics. Like the integrated economy, the world economy jumps onto or converges to a balanced-growth path. Each country produces the integrated-equilibrium quantities, X, of the domestically invented intermediates. Each country's share in the total number of intermediate goods markets, $a^m/A^{1-\mathcal{D}}$, converges to its share of total world-wide population, n^m/N.
(b) With multinational corporations or patent licensing (a'^m not necessarily equal to a^m), the equilibrium described in case (a) is one among several equilibria. Compared to this scenario, countries with low initial values for $a^m/A^{1-\mathcal{D}}$ can accelerate or decelerate the convergence process by moving production to foreign countries (such that $a'^m < a^m$) or attracting the production of intermediates invented abroad (such that $a'^m < a^m$), respectively.
Graphically, due to the absence of physical capital, the boxes in Figures 1 and 2 become lines, and $\bar{l}^1$ and $\bar{l}^2$ determine the endowment point on this line in a two-country world. Since final-goods production does not make use of labor, (a) there is only one production vector (viz., $(K_A, L_A)'$) if $a'^m = a^m$. That is why the division between countries is determinate. (b) The indeterminacy in the case of a'^m possibly different from a^m arises from the fact that there are two production vectors ($A^{1-\mathcal{D}}(K_X, L_X)'$ and $(K_A, L_A)'$) to be split up.
Examples 9 and 12: With $a_{K_Y} = a_{K_A} = 0$, $a_{K_X} = 1$, $a_{L_X} = 0$, and $\dot{l} = 0$, (22) and (23) become

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{L_Y} & a_{L_A} \end{pmatrix} \begin{pmatrix} y^m \\ A^{1-\chi}\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix} \tag{25}$$

and

$$\begin{pmatrix} \bar{k}^m \\ \bar{l}^m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ a_{L_Y} & 0 & a_{L_A} \end{pmatrix} \begin{pmatrix} y^m \\ a'^m X \\ A^{1-\chi}\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}}(i^m)^{\mathcal{D}} \end{pmatrix}, \tag{26}$$

[26] That is, this differential equation in $a^m/A^{1-\mathcal{D}}$ is autonomous in Example 8 and non-autonomous in Example 11.

respectively. Suppose (a) $a'^m = a^m$. It is obvious from (25) that (2) physical capital must then be mobile (i.e., $\bar{k}^m$ must be endogenous). Capital used in country m is determined by $a'^m = a^m$ and the definition of $\bar{k}^m$: $\bar{k}^m = a^m X$. If (b) a'^m is allowed to differ from a^m, we can allow for (3) immobility of physical capital. Then $\bar{k}^m$ is given, and a'^m adjusts such that $\bar{k}^m = a'^m X$ is satisfied (see the first equality in (26)). Given that either (a) $a'^m = a^m$ and (2) physical capital is mobile or (3) a'^m need not equal a^m, the second equality in (25) or (26), respectively, can be written as

$$\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}} (i^m)^{\mathcal{D}} = \frac{n^m l - a_{L_Y} y^m}{a_{L_A} A^{1-\chi}}.$$

Using $a_{L_Y} = L_Y/Y$ and $\dot{A} = L_A/(a_{L_A} A^{\chi})$ in the PV model and $I = L_A/(a_{L_A} A^{1-\chi})$ in the QU model, respectively, yields

$$\frac{d}{dt}\left(\frac{a^m}{A^{1-\mathcal{D}}}\right) = \frac{L}{a_{L_A} A^{1-\chi}}\left(\frac{n^m}{N} - \frac{y^m}{Y}\frac{L_Y}{L} - \frac{a^m}{A^{1-\mathcal{D}}}\frac{L_A}{L}\right).$$

In one equilibrium, $y^m/Y = a^m/A^{1-\mathcal{D}}$. In this case, the term in brackets on the right-hand side simplifies to $n^m/N - a^m/A^{1-\mathcal{D}}$. As the term in front of the brackets converges to a constant, country m's market share, $a^m/A^{1-\mathcal{D}}$, converges to its share of world-wide population, n^m/N. The transitional dynamics thus look as follows. The world economy converges to its balanced-growth path in the same way as the hypothetical integrated economy does. Each country's share of world-wide final goods production is proportional to its share of the number of innovations performed, with the quantity produced, X, identical everywhere, which presupposes that either physical capital flows or adjustments of a'^m ensure $\bar{k}^m = a'^m X$. Because factor prices equalize, l_Y^m/L_Y is equal to y^m/Y, which determines l_Y^m. As country m employs the remainder of its workforce, $l_A^m = \bar{l}^m - l_Y^m = \bar{l}^m - (y^m/Y)L_Y$, in R&D, it acquires a share of the number of intermediate goods markets equal to its share of world-wide population. Other equilibria exist: by decreasing y^m/Y below $a^m/A^{1-\mathcal{D}}$, thereby increasing R&D employment, $l_A^m = \bar{l}^m - (y^m/Y)L_Y$, a country with a low initial value $a^m/A^{1-\mathcal{D}}$ can accelerate the convergence process, and vice versa.

The two-country special case can be illustrated graphically. $\bar{l}^m \equiv n^m l \geq 0$ ($m \in \{1, 2\}$) is automatically satisfied. (a) The production of intermediates is the only use of capital and is not mobile internationally if $a'^m = a^m$. So the height of the box in Figure 1 is zero. Since there are two integrated-equilibrium production vectors (viz., $(K_Y, L_Y)'$ and $(K_A, L_A)'$), there are various ways of allocating world outputs to the individual countries. (b) With a'^m not necessarily equal to a^m, intermediate goods production can be shifted across borders, and there are three production vectors corresponding to internationally mobile economic activities: $A^{1-\mathcal{D}}(K_X, L_X)'$, $(K_Y, L_A)'$, and $(K_A, L_A)'$. The first is vertical while the other two are horizontal, because the intermediates are produced from physical capital alone and capital is not used in final-goods production or R&D. Thus, the hexagon formed by the integrated-equilibrium production vectors coincides with the box itself. (3) With physical capital immobile, $\bar{k}^m$

is determinate and positive: $\bar{k}^m \equiv k^m$ ($m \in \{1,2\}$). The endowment point $(\bar{k}^1, \bar{l}^1)$ uniquely determines the portions of $A^{1-\mathcal{D}}(K_X, L_X)'$ produced in the two countries. By contrast, numerous ways of dividing $(K_Y, L_A)'$ and $(K_A, L_A)'$ between the two countries exist. If (2) capital is mobile, there is another degree of freedom in splitting up the integrated-equilibrium production vectors, since the endowment point is not determinate.

Example 14: As (1) physical capital is absent, the conditions of Theorem 1 are satisfied if either (a) $a'^m = a^m$ and $\bar{l}^m \equiv n^m l - a_{L_X} X a'^m - n^m a_{l_l} \dot{l} \geq 0$ or (b) a'^m need not be equal to a^m. One of these conditions is assumed to be satisfied. The graphical illustration is analogous to Examples 8 and 11. From Theorem 2, since the integrated economy converges to its balanced-growth path, so does the world economy. From (21), we have

$$\left(\frac{\dot{a}^m}{A}\right)^{1-\mathcal{D}} (i^m)^{\mathcal{D}} = \frac{n^m l - a_{L_X} X a'^m - n^m a_{l_l} \dot{l}}{a_{L_A} A^{1-\chi}}.$$

(a) Let $a'^m = a^m$. Then, following the same steps as in Examples 8 and 11 and making use of $a_{l_l}\dot{l} = l_l$, we get

$$\frac{d}{dt}\left(\frac{a^m}{A^{1-\mathcal{D}}}\right) = \frac{L}{a_{L_A} A^{1-\chi}} \left(1 - \frac{l_l}{l}\right) \left(\frac{n^m}{N} - \frac{a^m}{A^{1-\mathcal{D}}}\right).$$

l_l/l converges towards its constant balanced-growth level. So by the same reasoning as in Example 8, country m's market share, $a^m/A^{1-\mathcal{D}}$, converges to its share of world population, n^m/N. (b) With a'^m not necessarily equal to a^m, there exists this equilibrium as well as others, with faster ($a'^m < a^m$), or slower ($a'^m > a^m$), convergence of countries with low initial values for $a^m/A^{1-\mathcal{D}}$.

5.8 International economic integration and long-run growth

If the assumptions for replication of the integrated equilibrium stated in Theorem 1 are satisfied, then, according to Theorem 3, the question of whether international trade and international knowledge spillovers boost long-run growth boils down to the question of whether the integrated equilibrium has the scale effect property. Unfortunately, the general model does not, of course, provide an unambiguous answer. In the learning-by-doing models, scale effects may (Example 3) or may not (Examples 2 and 4) be present. The human capital models without R&D (Examples 5-7) do not exhibit scale effects. R&D models with diminishing returns in R&D do not have the scale effect property either (Examples 8-10, 14, and 15). In R&D models with constant returns in R&D, growth is positively related to size (Examples 11-13). In view of the ambiguity of these theoretical results, the question becomes an empirical one. The empirical evidence on scale effects is also controversial. Backus, Kehoe, and Kehoe [8] present several regressions which cast doubt on the statistical significance and empirical relevance of scale effects. For instance, in a cross-section of countries they find that the effect of GDP on the growth rate of GDP per capita is insignificant. To give an impression of the size of the effect, they reckon that a hundred-fold increase in total GDP is associated with an increase in per capita growth of less than one percentage point ([8], p. 387). Similar results obtain with other scale variables. Kremer [30] objects, in line with our general model, that the relevant unit of analysis is not individual countries but geographical areas which share a common pool of technology. He argues that world-wide population growth as well as population growth in technologically separate regions from 1,000,000 B.C. to 1990 is consistent with a model in which technical change is proportional to the level of population (i.e., with scale effects) if one also adopts the Malthusian assumption that population is limited by technology. The latter assumption implies, however, that the model features constant GDP per capita in the long run. Moreover, the implied positive link between population and population growth has broken down in the more recent past. Jones [28] launched the most forceful attack on the scale effect hypothesis by pointing out that employment in R&D increased several-fold in the industrial nations in the post-war period without an accompanying boost in total factor productivity. Segerstrom [45] provides similar evidence. Barro and Sala-i-Martin [9] (p. 442) use the log of the working-age population as their measure of the economy's scale in their cross-country regression and report a positive but insignificant coefficient. Recently, Todo and Miyamoto [51] have argued that a more careful look at the data may bring about "the revival of scale effects". As yet, it seems fair to say that the presence of scale effects is at best controversial. This casts doubt on the proposition that international economic integration has an impact on the long-run growth rate.

6 Conclusion

In this paper, we have analyzed a fairly general open-economy growth model. Our main result is a Dixit-Norman theorem: under certain conditions, the world economy replicates the integrated equilibrium. Physical capital mobility, similarity of relative factor endowments, and the presence of multinational corporations or international patent licensing are favorable to the replicability of the integrated equilibrium. In addition to extending the Dixit-Norman approach to a new class of models, this result helps investigate the, as yet mostly unexplored, transitional dynamics of many open-economy growth models, such as the open-economy versions of the growth models of Solow [47], Romer [41], Lucas [35], Romer [42], Grossman and Helpman [24] (Chapters 3 and 4), Jones [27], and Segerstrom [45], among others. It also sheds light on the question of whether international economic integration promises long-run growth gains by linking it to the question of whether scale effects prevail in the integrated economy. It is hoped that the model provides a step in the direction of "a consistent body of theory that explains the relationship between trade and growth" Lewer and van den Berg [32] (p. 390) call for.

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Supplement: A "toolkit" for calculating growth rates

A.1 "Toolkit"

In this supplement, we provide a "toolkit" for calculating the rate of growth of aggregate output in a balanced-growth equilibrium. For most models, four equations are sufficient to calculate the growth rates listed in Section 5 of the paper. We denote the growth rate $\dot{z}/z$ of any variable z as g_z, and we usually suppress the arguments of the input coefficients functions, cost functions, and cost share functions. Here are the four equations and their proofs:

$$g_Y = \frac{[\theta_{L_Y}(1-\varepsilon) + \theta_{D_Y}\theta_{L_X}](g_l + g_N) + \theta_{D_Y}\gamma g_A}{1 - \theta_{K_Y} - \eta\theta_{L_Y} - \theta_{D_Y}\theta_{K_X}}. \tag{A.1}$$

$$\begin{cases} g_A = \frac{\theta_{K_A} g_Y + \theta_{L_A}(g_l + g_N)}{1-\chi}; & \chi < 1 \\ r + g_A - g_Y = \left(1 - \frac{1}{\mu}\right) \frac{\frac{L}{a_{L_A}} - g_A}{\frac{\theta_{L_Y}}{1-\theta_{K_Y}-\theta_{L_Y}} + \frac{\theta_{L_X}}{\mu}}; & \chi = 1 \end{cases} \tag{A.2}$$

$$g_Y - g_N = \frac{r - \rho}{\sigma} \tag{A.3}$$

$$g_Y - g_N - g_l = r - \frac{w}{c_l}. \tag{A.4}$$

Equation (A.1): From (1), $g_{K_Y} = g_Y$, and $g_{L_Y} = g_l + g_N$, we have

$$g_Y = \theta_{K_Y} g_Y + \theta_{L_Y}(g_{B_Y} + g_l + g_N) + \theta_{D_Y} g_{D_Y}.$$

From (2), $g_{K_Y} = g_Y$, and $g_{L_Y} = g_l + g_N$, it follows that

$$g_{B_Y} = \eta g_Y - \varepsilon(g_l + g_N).$$

In the PV model, from (3), $g_{D_Y} = g_A/\alpha + g_X$, and from (5), $g_X = \theta_{K_X}(g_Y - g_A) + \theta_{L_X}(g_l + g_N - g_A)$, where use is made of the fact that each of the A intermediate goods producers uses a fraction $1/A$ of the total resources in intermediate goods production. Taken together, it follows that

$$g_{D_Y} = \gamma g_A + \theta_{K_X} g_Y + \theta_{L_X}(g_l + g_N).$$

In the QU model, from (4), differentiating $\log D_Y = \log\lambda \int_0^1 \Omega(j)dj + \log X$ yields $g_{D_Y} = \log\lambda \cdot g_A + g_X$, where use is made of (7). Together with $g_X = \theta_{K_X} g_Y + \theta_{L_X}(g_l + g_N)$ (from (5)) and $\gamma \equiv \log\lambda$, we get the same equation for g_{D_Y} as in the PV case above. Taken together, the formulas for g_Y, g_{B_Y}, and g_{D_Y} yield (A.1).

Eq. (A.2): Suppose first $\chi < 1$. From (6), $g_A = F_A(K_A, L_A)/A^{1-\chi}$. So in a balanced-growth equilibrium with g_A constant, $\theta_{K_A} g_Y + \theta_{L_A}(g_l + g_N) = (1-\chi)g_A$, from which (A.2) follows immediately.

For $\chi = 1$, $g_A = F_A(K_A, L_A)$. For balanced growth, capital must not be an argument of the R&D technology and the labor supply must not rise. This requires $F_A(K_A, L_A) = L_A/a_{L_A}$, where a_{L_A} (> 0) is exogenous, $g_N = 0$, and $F_l(k_l, l_L, l'_l) = 0$, so that $\dot{l} = 0$ and the labor market clearing condition in (20) becomes

$$L = a_{L_Y} Y + a_{L_X} A^{1-\mathcal{D}} X + a_{L_A} g_A.$$

Differentiating the equation for the value of an innovation (14) and substituting for π from (12) gives the no-arbitrage condition $r + \mathcal{D} g_A - g_v = (1 - 1/\mu)pX/v$. Given $F_A(K_A, L_A) = L_A/a_{L_A}$, the free entry condition (15) becomes $A^{1-\mathcal{D}} v = w a_{L_A}$. Since the labor force is constant, the requirements that the growth rate of labor is identical in its different uses and that the proportion of labor income to Y is constant imply $g_w = g_Y$. Differentiating the free entry condition shows $(1 - \mathcal{D})g_A + g_v = g_Y$. Eliminating v and g_v from the no-arbitrage condition gives:

$$r + g_A - g_Y = \left(1 - \frac{1}{\mu}\right) \frac{pA^{1-\mathcal{D}} X}{w a_{L_A}}.$$

The pricing condition (9) can be written as $1 = r a_{K_Y} + w a_{L_Y} + pA^{1-\mathcal{D}} X/Y$ or

$$pA^{1-\mathcal{D}} X = \mu c_X A^{1-\mathcal{D}} X = (1 - \theta_{K_Y} - \theta_{L_Y})Y,$$

where use has been made of (11), $\theta_{K_Y} \equiv r a_{K_Y}$, and $\theta_{L_Y} \equiv w a_{L_Y}$. Eliminating $A^{1-\mathcal{D}} X$ from the labor market clearing condition and the no arbitrage equation yields

$$L = \left(a_{L_Y} + \theta_{L_X} \frac{1 - \theta_{K_Y} - \theta_{L_Y}}{\mu w}\right) Y + a_{L_A} g_A$$

(where use has been made of $a_{L_X}/c_X = \theta_{L_X}/w$) and

$$r + g_A - g_Y = \left(1 - \frac{1}{\mu}\right) \frac{1 - \theta_{K_Y} - \theta_{L_Y}}{w a_{L_A}} Y,$$

respectively. Substituting for Y from the former equation into the latter, using $w a_{L_Y} = \theta_{L_Y} c_Y = \theta_{L_Y}$, and rearranging terms proves the validity of the formula for the case $\chi = 1$ in (A.2).

Eq. (A.3): This is the Ramsey rule (16) with $g_c = g_Y - g_N$.

Eq. (A.4): The per capita cost of education, $c_l \dot{l}$, can be written as $c_l l g_l$. So in a balanced-growth equilibrium in which this cost accounts for a constant proportion of output per capita, $\dot{c}_l/c_l + g_l = g_Y - g_N$. Inserting this into (17) yields (A.4).

A.2 Models without human capital accumulation and without R&D

$F_l(k_l, l_l, l'_L)$ is identically zero ($g_l = 0$). $F_A(K_A, L_A)$ is identically zero. D_Y is deleted from the production function ($\theta_{D_Y} = 0$, $\theta_{K_Y} + \theta_{L_Y} = 1$). Since final goods production is the only use of capital and labor, we have $K_Y = K$, $L_Y = L$, and $Y = F_Y(K, B_Y L)$. Eq. (A.1) simplifies to:

$$g_Y = \frac{1-\varepsilon}{1-\eta} g_N. \tag{A.5}$$

Example 1: $\eta = \varepsilon = 0$. The formula in the main text follows immediately from (A.5).

Example 2: $0 < \eta < 1$, $\varepsilon = 0$. The formula in the main text follows immediately from (A.5).

Example 3 (Romer [41]): $\eta = 1$, $\varepsilon = 0$, $g_N = 0$. Eq. (A.5) is of no use here (it yields $0 = 0$). Profit maximization in the final goods sector implies $r = D_1 F_Y(K, B_Y L)$. Because of $B_Y = K_Y$ and the fact that the partial derivatives of a constant-returns-to-scale function are homogeneous of degree zero, $r = D_1 F_Y(1, L)$. From the Ramsey rule (A.3) with $g_N = 0$, one obtains the formula in the main text.

Example 4: $\eta = \varepsilon = 1$, $g_N = 0$. As in Example 3, $r = D_1 F_Y(K, B_Y L)$. With $B_Y = K_Y/L_Y$ and the homogeneity of degree zero of $D_1 F$, we have $r = D_1 F_Y(1, 1)$. Inserting this and $g_N = 0$ into the Ramsey rule (A.3) yields the formula in the main text.

A.3 Models with human capital accumulation and without R&D

$F_A(K_A, L_A)$ is identically zero, and $Y = F_Y(K_Y, B_Y L_Y)$ for simplicity ($\theta_{D_Y} = 0$, $\theta_{K_Y} + \theta_{L_Y} = 1$). Eq. (A.1) simplifies to:

$$g_Y = \frac{1-\varepsilon}{1-\eta}(g_l + g_N). \tag{A.6}$$

Example 5: $\eta = \varepsilon = 0$, $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$. The cost function associated with the simplified education technology is $c_l = w a_{l_l}$. So (A.4) can be written as $g_Y - g_N - g_l = r - 1/a_{l_l}$. Given the absence of externalities, (A.6) becomes $g_Y = g_l + g_N$. So the interest rate equals $r = 1/a_{l_l}$ and the Ramsey rule (A.3) yields the formula in the main text.

Example 6: $N = 1$, $\eta = 0$, $B_Y = L_Y^{-\varepsilon}$ ($\varepsilon < 0$). By the same reasoning as in Example 5, $g_Y - g_l = r - 1/a_{l_l}$. Together with (A.6), it follows that $-\varepsilon g_Y/(1-\varepsilon) = r - 1/a_{l_l}$. Eliminating r with the help of the Ramsey rule (A.3), we get the formula in the main text.

Example 7: $\eta = \varepsilon = 0$, so that $g_Y = g_l + g_N$. Capital and other people's labor are used in human capital formation. Because of constant returns to scale in education, the unit cost function $c_l(r, w)$ is linearly homogeneous. It follows that $w/c_l(r, w) = 1/c_l(r/w, 1)$. Inserting these two results into (A.4) yields

$$r = \frac{1}{c_l\left(\frac{r}{w}, 1\right)}.$$

A second equation that relates r to r/w is obtained from final goods production. Since $D_1 F_Y$ is homogeneous of degree zero, $r = D_1 F_Y(K_Y, L_Y) = D_1 F_Y(a_{K_Y}, a_{L_Y}) = D_1 F_Y(a_{K_Y}/a_{L_Y}, 1)$. Since the input coefficients are also homogeneous of degree zero, this can be written as:

$$r = D_1 F_Y\left[\frac{a_{K_Y}\left(\frac{r}{w}, 1\right)}{a_{L_Y}\left(\frac{r}{w}, 1\right)}, 1\right].$$

These two equations give continuous relationships between r and r/w. As r/w rises, so does $c_l(r/w, 1)$, so that r falls in the first equation. As r/w rises, a_{K_Y}/a_{L_Y} falls and $D_1F_Y(a_{K_Y}/a_{L_Y}, 1)$ increases. So if a solution $(r, r/w)$ to the pair of equations exists, it is unique, and a solution does exist for well-behaved neoclassical production functions F_Y. The steady-state growth rate is obtained by inserting the equilibrium interest rate, r, into the Ramsey rule (A.3).

From now on, $\eta = \varepsilon = 0$ without further mention.

A.4 Models with diminishing returns in R&D and without human capital accumulation

$\chi < 1$, $F_l(k_l, l_l, l_l')$ is identically zero ($g_l = 0$). From (A.1) and (A.2),

$$g_Y = \frac{(\theta_{L_Y} + \theta_{D_Y}\theta_{L_X})g_N + \theta_{D_Y}\gamma g_A}{1 - \theta_{K_Y} - \theta_{D_Y}\theta_{K_X}} \tag{A.7}$$

$$g_A = \frac{\theta_{K_A} g_Y + \theta_{L_A} g_N}{1 - \chi}. \tag{A.8}$$

Example 8: In Jones' PV model, $Y = L_Y^{1-\alpha} D_Y^{\alpha}$. Hence, $\theta_{L_Y} = 1 - \alpha$, $\theta_{K_Y} = 0$, and $\theta_{D_Y} = \alpha$. Moreover, $\theta_{K_X} = 1$, $\theta_{L_X} = 0$, $\theta_{L_A} = 1$, $\theta_{K_A} = 0$. Eq. (A.8) immediately gives the rate of innovation: $g_A = g_N/(1 - \chi)$. Together with (A.7), one obtains the expression for g_Y in the main text.

Example 9: In Segerstrom's QU model, $\theta_{D_Y} = 1$, $\theta_{L_Y} = \theta_{K_Y} = 0$, $\theta_{L_X} = 1$, $\theta_{K_X} = 0$, $\theta_{L_A} = 1$, $\theta_{K_A} = 0$. As in Example 8, $g_A = g_N/(1 - \chi)$ according to (A.8). Eq. (A.7) becomes $g_Y = g_N + \gamma g_A$. Substituting for g_A from the preceding equation gives the formula in the main text.

Example 10: F_Y, F_X, and F_A are Cobb-Douglas. Then the expression for g_Y is obtained by solving (A.7) and (A.8), given the θ's.

A.5 Models with constant returns in R&D and without human capital accumulation

$\chi = 1$, a_{L_A} is exogenous, $\theta_{L_A} = 1$, $\theta_{K_A} = 0$, $F_l(k_l, l_l, l_l')$ is identically zero ($g_l = 0$), and $g_N = 0$. Eq. (A.2) simplifies to

$$g_Y = \frac{\theta_{D_Y}\gamma g_A}{1 - \theta_{K_Y} - \theta_{D_Y}\theta_{K_X}}. \tag{A.9}$$

Example 11: In Romer's PV model, $Y = L_Y^{1-\alpha} D_Y^{\alpha}$ ($\theta_{L_Y} = 1 - \alpha$, $\theta_{K_Y} = 0$, $\theta_{D_Y} = \alpha$), $\theta_{K_X} = 1$, and $\theta_{L_X} = 0$. From (A.2) and (A.9),

$$r + g_A - g_Y = \left(1 - \frac{1}{\mu}\right) \frac{\frac{L}{a_{L_A}} - g_A}{\frac{1-\alpha}{\alpha}}$$

and

$$g_Y = \frac{\alpha\gamma g_A}{1-\alpha},$$

respectively. Using the latter equation and the Ramsey rule (A.3) to eliminate g_A and r from the former equation, we get the formula in the main text.

Example 12: $\theta_{D_Y} = 1$, $\theta_{L_Y} = \theta_{K_Y} = 0$, $\theta_{L_X} = 1$, $\theta_{K_X} = 0$. From (A.2) and (A.9),

$$r + g_A - g_Y = (\mu - 1)\left(\frac{L}{a_{L_A}} - g_A\right)$$

and

$$g_Y = \gamma g_A,$$

respectively. With the help of (A.3), we obtain the formula for g_Y in the main text.

Example 13: F_Y and F_X are Cobb-Douglas. The expression for g_Y is obtained by solving (A.2), (A.3), and (A.9), for given θ's.

A.6 Models with diminishing returns in R&D and with human capital accumulation

Example 14: $\chi < 1$, $\theta_{D_Y} = 1$, $\theta_{L_Y} = \theta_{K_Y} = 0$, $\theta_{L_X} = 1$, $\theta_{K_X} = 0$, $\theta_{L_A} = 1$, $\theta_{K_A} = 0$, $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$, and $g_N = 0$. As in Example 5, $w/c_l = 1/a_{l_l}$, so (A.4) becomes $g_Y - g_l = r - 1/a_{l_l}$. From (A.1), eliminating r with the help of (A.3) gives

$$g_l = \frac{1}{a_{l_l}} - \rho - (\sigma - 1)g_Y.$$

Eqs. (A.1) and (A.2) imply $g_Y = g_l + \gamma g_A$ and $g_A = g_l/(1-\chi)$, respectively. Hence

$$g_Y = \left(1 + \frac{\gamma}{1-\chi}\right) g_l.$$

Substituting for g_l from the former equation into the latter proves the validity of the formula for g_Y in the main text.

Example 15: F_Y, F_X, and F_A are Cobb-Douglas and $F_l(k_l, l_l, l_l') = l_l/a_{l_l}$. The growth rate of aggregate output is obtained following the same steps as in the preceding example.